Lecture 24

In addition to revealing how the electric field of a point charge becomes anisotropic when it is moving (causing radiation fields to form when the motion changes), the Lorentz transformation rule for the electric field is important for quite another thing entirely.

Recall that conceptually the electric field began as "the local entity responsible for the force on a charged particle". At the time we didn't concern ourselves with the state of motion of the particle.
that experiences the force. However, our assumption was clearly that the charge was at rest initially or at least moving slowly (compared with $c$). But now that we know how electric fields transform, we can go into the instantaneous rest frame of even a rapidly moving charge and work out the force there, then transforming everything back to the original frame.

So here's the scenario. In the "lab" frame (unprimed) we have a uniform $\vec{E}$ and a charge $q$ whose velocity at time $t=0$ is $\vec{V}$.
The momentum of the charge is \( \vec{p} \) which we decompose as \( \vec{p} = \vec{p}_\parallel + \vec{p}_\perp \), where \( \vec{p}_\parallel \) is parallel to the instantaneous velocity \( \vec{v} \) at \( t=0 \). The electric field is decomposed according to the same axis defined by \( \vec{v} \) into \( \vec{E} = \vec{E}_\parallel + \vec{E}_\perp \). When we go into the "particle" frame (primed) we see a transformed electric field: \( \vec{E}_\prime = \vec{E}_\parallel', \vec{E}_\prime = \gamma \vec{E}_\perp \).

In the lab frame we would like to know:

\[
\frac{d\vec{p}_\parallel}{dt} = \ldots
\]

\[
\frac{d\vec{p}_\perp}{dt} = \ldots
\]
at time $t=0$. We have omitted the arrow over $\mathbf{p}$ since its direction is defined by $\mathbf{v}$.

Using the Lorentz transformation we will transform the differentials in these expressions into the primed frame where the particle is instantaneously at rest.

Recall that $\mathbf{p}$, together with the particle energy $E/c$, forms a 4-vector with transformation

$$\mathbf{p'} = \gamma (\mathbf{p} + \frac{\mathbf{v}}{c^2} E')$$

$$\overrightarrow{\mathbf{p}}_1 = \overrightarrow{\mathbf{p}}_1' \quad \nu = |\mathbf{v}|$$

The same transformation applies to the differentials (differences of 4-vectors):
\[ dp_{||} = \gamma (dp_{||} + \frac{2v}{c^2} de') \]
\[ dp_{\perp} = dp_{\perp}' \]

The energy and momentum magnitude satisfy the invariant-rest-mass relation

\[ (\varepsilon')^2 = (p'c)^2 + (mc^2)^2 \]

from which we obtain

\[ \varepsilon' de' = c^2 p'dp' \]

But in the instantaneous rest frame \( p' = 0 \), so \( de' = 0 \). This corresponds, in non-relativistic language, to the fact that the power given to the particle in a force field vanishes at times
when the particle is at rest:  
\[ \text{power} = \frac{de'}{dt'} = \vec{v}' \cdot \vec{F} \]

\[ \vec{v}' = 0 \Rightarrow de' = 0. \]

We thus get the simplification

\[ dp_{11} = \gamma dp'_{11} \]
\[ \vec{dp}_1' = \vec{dp}_1' \]

The differential \( dp \) corresponds to a momentum difference between two events on the particle's world line. Let's examine the relation between these (closely spaced) events in the two frames.
The proper time $d\tau$ between events $A$ and $B$ is the same in any frame and equals the coordinate time $dt'$ in the frame where the particle is at rest:

$$d\tau = dt' = \sqrt{dt'^2 - (\frac{v}{c})^2 dt^2}$$

$$= \frac{dt}{\gamma}$$

We use this relation between $dt'$ and $dt$ to transform our momentum
derivatives:

\[
\frac{dP_i}{dt} = \gamma \frac{dP_i'}{dt'} = \frac{dP_i'}{dt'} = qE_i' = \frac{dE_i}{dt} \quad \text{(force law in rest frame)}
\]

\[
\Rightarrow \quad \frac{dP_i}{dt} = qE_i
\]

\[
\frac{d\vec{P}_i}{dt} = \frac{d\vec{P}_i'}{dt'} = \frac{1}{\gamma} \left( \frac{d\vec{P}_i'}{dt'} \right) \quad \text{(force law)}
\]

\[
= \frac{1}{\gamma} (qE_i') = qE_i
\]

\[
\Rightarrow \quad \frac{d\vec{P}_i}{dt} = q\vec{E}_i
\]

The content of these two laws,
which can be combined into a single vector equation,

\[
\frac{d\vec{P}}{dt} = q\vec{E},
\]

is actually quite simple. The rate of momentum change, as calculated in any frame, is just \( q \) times the electric field in that frame.