Lecture 24: March 27

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24.1 Time-dependent canonical transformations

In lecture 23 we developed the "generating function" method of constructing canonical transformations. In that lecture we treated the time t in the formula for the transformation as a fixed parameter that was unrelated to the t in the time evolution of Hamilton's equations. Because the transformation produced by the function F is canonical, the form of Hamilton's equations is preserved.

We can no longer assume that the form of Hamilton's equations are unchanged when we allow the t parameter in the generating function (and corresponding canonical transformation) to change in the course of time evolution. To address this problem we revisit the derivation of Hamilton's equations from the extremal action principle.

24.1.1 Canonical transformations and Hamilton's principle

Recall that Hamilton's principle, now for a single degree of freedom, asserts that the functional

$$S[q,p] = \int_{t_1}^{t_2} (p\,\dot{q} - \mathcal{H})\,dt, \tag{24.1}$$

is extremized for trajectories $q^\star(t)$, $p^\star(t)$ that satisfy Hamilton's equations. This is true even when the Hamiltonian has direct time dependence. To investigate what happens when we perform a time dependent canonical transformation, we start by rewriting the integral above in terms of the transformed variables:

$$\begin{aligned} Q &= Q(q,p,t) && (24.2)\\ P &= P(q,p,t). && (24.3) \end{aligned}$$

Note that

$$p\,\dot{q}\,dt = p\,dq \tag{24.4}$$

is a line element in the (q,p) plane. In lecture 23 we learned how to express this line element as

$$p\,dq = P\,dQ|_t + dF|_t, \tag{24.5}$$

where

$$\begin{aligned} dQ|_t &= \frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp && (24.6)\\ dF|_t &= \frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial Q}dQ|_t && (24.7) \end{aligned}$$

are the differentials of Q and the generating function F at fixed time. We would like to be able to replace $dQ|_t$ by dQ, so that $P\,dQ = P\,\dot{Q}\,dt$ has the correct form for the action principle in the (Q, P) variables. Similarly, we would like to replace $dF|_t$ by dF, so that its integral just gives irrelevant endpoint terms.

Replacing $dQ|_t$ is straightforward since

$$dQ|_t = dQ - \frac{\partial Q}{\partial t}dt. \tag{24.8}$$

Before we try to replace $dF|_t$ we should recall how the generating function was defined:

$$F = F(q, Q(q, p, t), t). \tag{24.9}$$

From this we see that F has direct time dependence by its third argument as well as indirectly through the transformed variable Q. Therefore,

$$\begin{aligned} dF|_t &= dF - \left(\frac{\partial F}{\partial Q}\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial t}\right)dt & (24.10)\\ &= dF + P\frac{\partial Q}{\partial t}dt - \frac{\partial F}{\partial t}dt, & (24.11)\end{aligned}$$

where we have used the equation from lecture 23, $-P = \partial F/\partial Q$. Substituting (24.8) and (24.11) into the action integral (24.1), we obtain

$$S = \int_{t_1}^{t_2}\left(P\,\dot{Q} - \left(\mathcal{H} + \frac{\partial F}{\partial t}\right)\right)dt \;+\; F(q, Q, t)\big|_{t_1}^{t_2}. \tag{24.12}$$

Since the variations of the coordinates δq and δQ are constrained to vanish at the endpoints of the integral, the second term in (24.12) always has zero variation and may be ignored. We therefore find that the form of the action principle — and hence Hamilton's equations — is preserved provided we transform the Hamiltonian as:

$$\mathcal{H}'(Q, P, t) = \mathcal{H}(q, p, t) + \frac{\partial}{\partial t}F(q, Q(q, p, t), t). \tag{24.13}$$

Consequently, when the canonical transformation is time-dependent it is not enough to simply express $\mathcal{H}$ in terms of the new variables: there is an additive correction as well (the term $\partial F/\partial t$).

Let's revisit the time-dependent transformation (lecture 23) generated by

$$F(q, Q, t) = M\frac{qQ}{t} \tag{24.14}$$

which gave

$$\begin{aligned} Q &= \frac{p}{M}t & (24.15)\\ P &= -M\frac{q}{t}. & (24.16)\end{aligned}$$

We will apply this transformation to the free-particle Hamiltonian in the original variables:

$$\mathcal{H} = \frac{p^2}{2M}. \tag{24.17}$$

The transformation parameter M in F coincides with the particle mass M, but this was our choice. By (24.13) the transformed Hamiltonian is

$$\begin{aligned} \mathcal{H}' &= \frac{p^2}{2M} - M\frac{qQ}{t^2} & (24.18)\\ &= \frac{1}{2}M\left(\frac{Q}{t}\right)^2 + \frac{PQ}{t}. & (24.19)\end{aligned}$$

This is a strange Hamiltonian indeed, and clearly no improvement over the original time-independent $\mathcal{H}$! Still, the Hamiltonian equations we get are easily solved and serve as a check of our derivation. Here are Hamilton's equations for the new variables,

$$\begin{aligned} \dot{Q} &= +\frac{\partial \mathcal{H}'}{\partial P} = \frac{Q}{t} && (24.20) \\ \dot{P} &= -\frac{\partial \mathcal{H}'}{\partial Q} = -M\frac{Q}{t^2} - \frac{P}{t}, && (24.21) \end{aligned}$$

and their most general solution:

$$\begin{aligned} Q(t) &= v_0 t && (24.22) \\ P(t) &= -Mv_0 - M\frac{q_0}{t}. && (24.23) \end{aligned}$$

This is the expected transformation of the most general free particle motion:

$$\begin{aligned} q(t) &= v_0 t + q_0 && (24.24) \\ p(t) &= Mv_0. && (24.25) \end{aligned}$$

A much more common application of time-dependent canonical transformations, one we will see in a future lecture, is to *remove* time-dependence from a Hamiltonian.

24.1.2 Four kinds of generating function

Because the phase-space area line integral can be expressed in two ways,

$$\oint_c p\,dq = \oint_c (-q\,dp), \tag{24.26}$$

there are altogether four ways we could have defined generating functions of which (24.9) and its consequence (24.5) was just the first:

$$\begin{aligned} +p\,dq &= +P\,dQ|_t + dF_1|_t && (24.27) \\ -q\,dp &= +P\,dQ|_t + dF_2|_t && (24.28) \\ +p\,dq &= -Q\,dP|_t + dF_3|_t && (24.29) \\ -q\,dp &= -Q\,dP|_t + dF_4|_t. && (24.30) \end{aligned}$$

The corresponding equations for the canonical transformations are listed below:

$$\begin{aligned} F_1 &= F_1(q,Q,t) & \frac{\partial F_1}{\partial q} &= +p & \frac{\partial F_1}{\partial Q} &= -P && (24.31) \\ F_2 &= F_2(p,Q,t) & \frac{\partial F_2}{\partial p} &= -q & \frac{\partial F_2}{\partial Q} &= -P && (24.32) \\ F_3 &= F_3(q,P,t) & \frac{\partial F_3}{\partial q} &= +p & \frac{\partial F_3}{\partial P} &= +Q && (24.33) \\ F_4 &= F_4(p,P,t) & \frac{\partial F_4}{\partial p} &= -q & \frac{\partial F_4}{\partial P} &= +Q. && (24.34) \end{aligned}$$

The rule (24.13) for transforming the Hamiltonian is the same for all four kinds of generating function.