Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 23: March 24

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23.1 Time-dependent canonical transformations

In the next two lectures we will develop a method for constructing time-dependent canonical transformations. We will need this tool later, when we study adiabatic invariance. As in the previous lecture we consider a single degree of freedom with a phase space of only two dimensions, so that we can make diagrams. The tools we develop, called "generating functions", generalize to multiple degrees of freedom, though we will stop short of doing that.

23.1.1 Canonical transformations at fixed time

We start by viewing time as a fixed parameter that appears in the transformation rule for the phase space variables:

$$Q = Q(q,p,t) \tag{23.1}$$
$$P = P(q,p,t). \tag{23.2}$$

As we learned in lecture 22, the transformation is canonical — it defines a proper new pair of phase space variables — when the Jacobian of the transformation is unity.

Now consider a region in the (q,p) plane enclosed by a simple (non-intersecting) closed boundary curve c. The image of c and its interior, under the transformation (23.1), is a simple closed curve C enclosing a region in the (Q,P) plane. Because the transformation is canonical, the two regions enclose exactly the same phase space areas.

A mathematical statement of the equality of areas is the equality of the corresponding line integrals:

$$\oint_c p\,dq = \oint_C P\,dQ. \tag{23.3}$$

We can re-express this as an identity on a line integral just on the curve c in the (q,p) plane,

$$\oint_c \left(p\,dq - P(q,p,t)\,dQ|_t\right) = 0, \tag{23.4}$$

where the notation $\cdots|_t$ reminds us that t is held fixed when we express the line element dQ in terms of dq and dp[1]:

$$dQ|_t = \frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp. \tag{23.5}$$

[1] This will be modified later, when we study time evolution.

The identity (23.4) holds for arbitrary simple closed curves c only if the integrand is a perfect differential in the variables q and p:

$$p\,dq - P\,dQ|_t = dF|_t. \tag{23.6}$$

It turns out to be expedient to define the arbitrary phase-space function F as a function of three variables as follows:

$$F = F(q, Q(q,p,t), t). \tag{23.7}$$

The differential of F, with t held fixed, is

$$dF|_t = \frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial Q}dQ|_t. \tag{23.8}$$

Since $dQ|_t$ is independent of dq (because the former also depends on dp), for (23.6) and (23.8) to be consistent we must have

$$p = \frac{\partial F}{\partial q} \tag{23.9}$$

$$-P = \frac{\partial F}{\partial Q}. \tag{23.10}$$

What we can now do is exploit the arbitrariness of functions F having arguments of the form (23.7) to design equations (23.9) & (23.10) of our choosing. These equations relate (q,p) and (Q,P), and the relationship must be canonical because the derivation imposed equality (23.3) of phase-space areas. The function F is called a "generating function" because it is being used to generate a canonical transformation.

As a simple example, take the function

$$F(q,Q,t) = M\frac{qQ}{t}. \tag{23.11}$$

Recall that dF is a phase-space area element and therefore has units of action. If q and Q are position coordinates with units of length, then the constant M would have units of mass in order that F has units of action. Here are the equations (23.9) & (23.10) for this particular generating function:

$$p = M\frac{Q}{t} \tag{23.12}$$

$$-P = M\frac{q}{t}. \tag{23.13}$$

Solving these algebraic equations for (Q,P) in terms of (q,p),

$$Q = \frac{p}{M}t \tag{23.14}$$

$$P = -M\frac{q}{t}, \tag{23.15}$$

gives us a canonical transformation.

The only way that this construction of a canonical transformation can fail is if the equations (23.9) & (23.10) do not uniquely define (Q,P) in terms of (q,p), *i.e.* the transformation is singular. This happens, for example, when $F = f(q) + g(Q)$ since then

$$p = f'(q) \tag{23.16}$$

$$-P = g'(Q) \tag{23.17}$$

imply that the phase-space variables (q,p) are not independent and do not determine (Q,P). To get a general necessary condition on F, we note that

$$1 = \frac{\partial p}{\partial p} = \frac{\partial}{\partial p}\left(\frac{\partial F}{\partial q}\right) = \left(\frac{\partial^2 F}{\partial Q \partial q}\right)\frac{\partial Q}{\partial p}, \tag{23.18}$$

so the mixed partial derivative of F may not vanish — as was the case in the example above.