Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 22: March 22

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22.1 Hamilton's principle in phase space

Hamilton's extremal action principle can be expressed in terms of the Hamiltonian variables — the q's and p's. For a system with N degrees of freedom, trajectories in phase space are described by $2N$ functions of time:

$$q_1(t), \ldots, q_N(t)\ ;\ p_1(t), \ldots, p_N(t) \tag{22.1}$$

We define an action functional

$$S[q_1, \ldots q_N\ ;\ p_1, \ldots, p_N] = \int_{t_1}^{t_2} \left(\sum_{k=1}^{N} \dot{q}_k p_k - \mathcal{H} \right) dt, \tag{22.2}$$

that treats all $2N$ functions as independent. We recognize the integrand as the Lagrangian. It is an easy exercise (assigned as homework) to show that the variation of this S vanishes to first order in the independent variations of the trajectory, when the trajectory satisfies Hamilton's equations of motion. The Hamiltonian action principle will be very useful later in the course, when we try to generate the types of transformations introduced in the next section.

22.1.1 Canonical transformations

In the Lagrangian formalism we often made use of the fact that we have enormous freedom in how a given system may be described by generalized coordinates of our choosing. For example, we might have a one-degree-of-freedom system with generalized coordinate q and Lagrangian $\mathcal{L}(q, \dot{q}, t)$. The equations of motion are

$$0 = \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right). \tag{22.3}$$

But suppose we find a more convenient coordinate

$$Q = Q(q, t) \tag{22.4}$$

that describes the same system. This represents a coordinate transformation. From what we have learned, the equations of motion for the transformed Lagrangian $\mathcal{L}'$ defined by

$$\mathcal{L}'(Q, \dot{Q}, t) = \mathcal{L}(q, \dot{q}, t), \tag{22.5}$$

should have exactly the same form:

$$0 = \frac{\partial \mathcal{L}'}{\partial Q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{Q}} \right). \tag{22.6}$$

We can try a similar, very general variable transformation (again for $N = 1$) of the Hamiltonian variables,

$$Q = Q(q, p, t) \tag{22.7}$$
$$P = P(q, p, t), \tag{22.8}$$

and ask if the form of Hamilton's equations is preserved. We will see, in the simplest case of time-independent transformations, that this is not true unless the transformation satisfies a particular property. Moreover, when the transformation has time-dependence we will find that a restriction of the transformation is not enough. In that case the Hamiltonian has to be transformed beyond simply substituting new variables for the old variables! As a warm-up for these questions we will go through the exercise of confirming the invariance of the form of the Euler-Lagrange equations, where we only transform the generalized coordinates.

Given equation (22.3), the definition of the transformed coordinate (22.4), and the transformed Lagrangian (22.5), our goal is to derive equation (22.6). This is mostly an exercise in the disciplined use of the chain rule of calculus. From (22.4) we infer

$$\dot{Q} = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial t}. \tag{22.9}$$

Since

$$\mathcal{L}(q, \dot{q}, t) = \mathcal{L}'(Q(q, t), \dot{Q}(q, \dot{q}, t), t), \tag{22.10}$$

and only the second argument of $\mathcal{L}'$ depends on $\dot{q}$, we obtain

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}'}{\partial \dot{Q}} \frac{\partial \dot{Q}}{\partial \dot{q}} \tag{22.11}$$
$$= \frac{\partial \mathcal{L}'}{\partial \dot{Q}} \frac{\partial Q}{\partial q}, \tag{22.12}$$

where we made use of (22.9). We get a more complicated result for the other partial derivative because both the first and second arguments of $\mathcal{L}'$ depend on q:

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{\partial \mathcal{L}'}{\partial Q} \frac{\partial Q}{\partial q} + \frac{\partial \mathcal{L}'}{\partial \dot{Q}} \frac{\partial \dot{Q}}{\partial q} \tag{22.13}$$
$$= \frac{\partial \mathcal{L}'}{\partial Q} \frac{\partial Q}{\partial q} + \frac{\partial \mathcal{L}'}{\partial \dot{Q}} \left(\frac{\partial^2 Q}{\partial q^2}\dot{q} + \frac{\partial^2 Q}{\partial q \partial t} \right). \tag{22.14}$$

For the Euler-Lagrange equation we need the time derivative of (22.11):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{Q}}\right) \frac{\partial Q}{\partial q} + \frac{\partial \mathcal{L}'}{\partial \dot{Q}} \frac{d}{dt}\left(\frac{\partial Q}{\partial q}\right) \tag{22.15}$$
$$= \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{Q}}\right) \frac{\partial Q}{\partial q} + \frac{\partial \mathcal{L}'}{\partial \dot{Q}} \left(\frac{\partial^2 Q}{\partial q^2}\dot{q} + \frac{\partial^2 Q}{\partial q \partial t} \right). \tag{22.16}$$

Subtracting (22.16) from (22.14) we obtain the transformed Euler-Lagrange combination,

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}'}{\partial Q} \frac{\partial Q}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{Q}}\right) \frac{\partial Q}{\partial q} \tag{22.17}$$
$$= \left(\frac{\partial \mathcal{L}'}{\partial Q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{Q}}\right)\right) \frac{\partial Q}{\partial q} \tag{22.18}$$

as a multiple of the original combination. Excepting special points where $\partial Q/\partial q$ vanishes, the transformed Euler-Lagrange equation must be satisfied (the combination equals zero) because the original equation was satisfied.

By contrast with the Lagrangian coordinate transformation (22.4), nothing as general as the phase space transformation (22.7) will automatically preserve the form of Hamilton's equations. We will start with the easiest case, where the transformation is time-independent:

$$Q = Q(q,p) \tag{22.19}$$
$$P = P(q,p). \tag{22.20}$$

Checking whether this preserves the form of Hamilton's equations starts with the definition of the transformed Hamiltonian,

$$\mathcal{H}(q,p) = \mathcal{H}'(Q(q,p), P(q,p)), \tag{22.21}$$

and applies the chain rule of calculus and algebraic manipulations (assigned in homework) to show that

$$\dot{Q} = +\frac{\partial \mathcal{H}'}{\partial P} \tag{22.22}$$
$$\dot{P} = -\frac{\partial \mathcal{H}'}{\partial Q}. \tag{22.23}$$

However, the derivation goes through only when the variable transformation satisfies the following property:

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \det \begin{bmatrix} \partial Q/\partial q & \partial P/\partial q \\ \partial Q/\partial p & \partial P/\partial p \end{bmatrix} = 1. \tag{22.24}$$

This is a condition on the Jacobian of the transformation. A value of unity implies that the transformation is locally area-preserving. In mechanics the term "canonical" is used for this property. The unit-Jacobian/canonical property came up in our discussion of Liouville's theorem in lecture 19. There we found that the transformation of phase space defined by time-evolution over a time interval Δt is area-preserving as $\Delta t \to 0$. Time evolution is but one example of a canonical transformation (details assigned as homework). There are also canonical transformations that have nothing to do with time evolution. As an example, we consider a transformation of the Hamiltonian for a falling body.

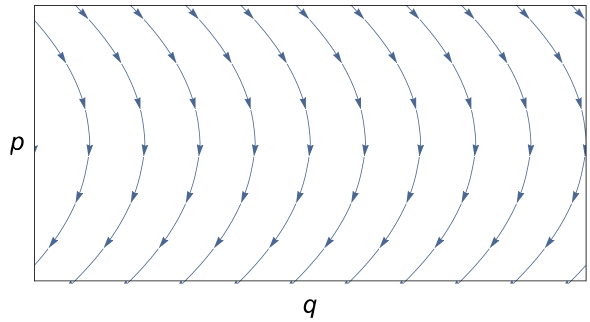
Let q be the height of the body (usually denoted z) and p the momentum conjugate to q (the z-component of linear momentum). For a body of mass M, the Hamiltonian is

$$\mathcal{H}(q,p) = \frac{p^2}{2M} + Mgq. \tag{22.25}$$

The Hamiltonian equations of motion are

$$\dot{q} = +\frac{\partial \mathcal{H}}{\partial p} = \frac{p}{M} \tag{22.26}$$
$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -Mg, \tag{22.27}$$

and produce a parabolic pattern of flow in phase space (next page).



The following transformation,

$$Q = q + \frac{p^2}{2M^2 g} \tag{22.28}$$
$$P = p, \tag{22.29}$$

can be checked to be canonical by computing the Jacobian. Since

$$\mathcal{H}(q,p) = \frac{p^2}{2M} + Mgq = Mg\,Q, \tag{22.30}$$

the transformed Hamiltonian takes the ludicrously simple form:

$$\mathcal{H}'(Q,P) = Mg\,Q. \tag{22.31}$$

As a result, the transformed Hamilton's equations

$$\dot{Q} = +\frac{\partial \mathcal{H}'}{\partial P} = 0 \tag{22.32}$$
$$\dot{P} = -\frac{\partial \mathcal{H}'}{\partial Q} = -Mg, \tag{22.33}$$

are so simple they can be solved by inspection:

$$Q = Q_0 \tag{22.34}$$
$$P = -Mg\,t + P_0. \tag{22.35}$$

The canonical transformation (22.28) & (22.29) had the effect of straightening out the flow field: