Physics 3318: Analytical Mechanics **Spring 2017**

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21.1 Phase space mixing

21.1.1 Ergodicity and statistical mechanics

While the rigor of Poincaré's recurrence theorem is unassailable, the statement it proves is not very useful. We are told there exists a phase space point $z(0)$ in a region $A(0)$ — that we are free to make as small as we like — and a time t such that the time-evolved point $z(t)$ will also be inside $A(0)$. Not only do we lack a good estimate of the minimum recurrence time t, we have no idea how plentiful points such as $z(0)$ are. If it turned out that "recurring" initial conditions are extremely rare, then there would not be much practical significance to the theorem.

It is actually a good thing that recurring initial points are extremely rare, otherwise the modeling of the properties of matter with *statistical mechanics* would not be possible. This subdiscipline of physics rests on a mathematical hypothesis about mechanical systems that is very difficult to prove, but is generally believed to be true. Although not a theorem, the *ergodic hypothesis* is more useful than the recurrence theorem because it has practical consequences.

The ergodic hypothesis asserts that if a mechanical system is sufficiently "mixing", then the time evolution $z(t)$ of an arbitrary initial phase space point $z(0)$, given sufficient time t, will come arbitrarily close to *every* accessible point in phase space. The "mixing" condition is technical and beyond the scope of these lectures, but a physical analogy and a simple model later in the lecture should give you a sense of this term.

Consider a bucket of white paint into which a small drop of black paint has been added. The paints are thick and remain separate, even when stirred with a stick. Phase space corresponds to the contents of the bucket, the evolving region of initial conditions $A(t)$ is the drop of black paint, and Hamiltonian evolution is the volume preserving flow of paint — crudely implemented by the moving stick. When the stick-motion/Hamiltonian-flow is sufficiently "mixing", then the paint will turn homogeneously gray on large scales. Though still perfectly segregated from the white paint on microscopic scales, the black drop will become increasingly stretched and filamentary, folding on itself into a structure that fills the entire bucket.

What does the ergodic hypothesis imply for the recurrence theorem? Assuming the time evolution of our system is "mixing up" phase space, the initial region $A(0)$ will become increasingly dispersed over all of phase space, its content stretched into thin filaments that have a uniform density. The fraction of initial conditions in $A(0)$ that return to $A(0)$ after a long time t corresponds to the fraction of the filamentary, phase-space filling region $A(t)$ that intersects $A(0)$. By the uniformity property of the ergodic hypothesis,

$$\text{fraction of recurring initial conditions in } A(0) = \frac{\text{vol}(A(0))}{\text{vol}(\Omega)}, \tag{21.1}$$

where Ω represents the accessible region of phase space, usually restricted only by having the same range of $\mathcal{H}$ values (initial energies) as the points in $A(0)$. The more precise we make the initial conditions, the smaller the fraction above becomes!

The ergodic hypothesis makes time evolution unnecessary when answering questions that are posed as time-averages. For suppose that $F(q_1(t), \ldots; p_1(t), \ldots)$ is a function defined by our system coordinates and momenta whose time average we wish to know. By the ergodic hypothesis, the point $(q_1(t), \ldots; p_1(t), \ldots)$ visits over the course of time every subvolume element $dq_1 \ldots dp_1 \ldots$ of the accessible phase space Ω with equal frequency. The time average is therefore the same as the phase space average:

$$\langle F(q_1(t), \ldots; p_1(t), \ldots)\rangle_t = \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega dq_1 \ldots dp_1 \ldots \; F(q_1, \ldots; p_1, \ldots). \tag{21.2}$$

The applicability of statistical mechanics and thermodynamics rests upon the validity of the expression above, since the time averaged properties of materials measured by experiments are recast as phase space averages. We only have proofs of the ergodic hypothesis for very simple systems, such as Sinai's "billiards": a system of two hard, elastic disks. Extensive numerical simulations and the internal consistency of thermodynamic measurements give overwhelming support to the hypothesis.

21.1.2 Chaos inside the proton

Statistical mechanics is often incorrectly and unfairly characterized as a form of analysis that applies only to systems with many degrees of freedom. But we made no reference to such a condition above, when we discussed the ergodic hypothesis, and in fact systems with as few as two degrees of freedom can display mixing behavior and have time averages that are exactly given by phase space averages. Small systems with mixing dynamics are usually referred to as "chaotic". We will describe such a system that exists in the interior of protons and neutrons.

Before we describe the analogues of electric and magnetic fields that bind the quarks inside the proton, we should review the dynamics of ordinary electric and magnetic fields. These do *not* have chaotic dynamics and the ergodic hypothesis therefore does not apply to them.

To keep things simple, we consider fields in the absence of charges and in a world with only two spatial dimensions. The electric field has two components, E_x and E_y, and the magnetic field just one, B. In the Coulomb gauge these are expressed in terms of the two components of the vector potential (in units where $c = 1$):

$$E_x = \dot{A}_x, \qquad E_y = \dot{A}_y, \qquad B = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}. \tag{21.3}$$

The vector potential corresponds to the generalized coordinates and the Lagrangian of the system is written in terms of it as follows:

$$\mathcal{L} = \int dx\, dy\, (E_x^2 + E_y^2 - B^2) \tag{21.4}$$

$$= \int dx\, dy \left(\dot{A}_x^2 + \dot{A}_y^2 - \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)^2 \right). \tag{21.5}$$

To further simplify things, suppose we have spatially uniform initial conditions. That is, we consider a system where initially A_x, A_y, $\dot{A}_x$, and $\dot{A}_y$ are constant in space. Spatial uniformity will be maintained at later times and therefore $B = 0$ at all times. The Lagrangian reduces to

$$\mathcal{L} = \frac{1}{2} M(\dot{A}_x^2 + \dot{A}_y^2), \tag{21.6}$$

where M is a constant (proportional to the area of space). This is the same as the Lagrangian of a free particle moving in two dimensions. The most general solution is

$$E_x = \dot{A}_x = \text{constant} \qquad E_y = \dot{A}_y = \text{constant}. \tag{21.7}$$

Not surprisingly, the most general, spatially uniform solution (in the absence of charges) is a uniform (in space and time) electric field and vanishing magnetic field. Clearly the behavior of this system is poorly mixing and time evolution is far from ergodic!

The analogs of electric and magnetic fields that glue together the quarks inside the proton are the Yang-Mills fields. These differ from the ordinary E_x, E_y, and B mostly by coming in three types of "color". Here they are, written in terms of the vector potential:

$$\mathbf{E}_x = \dot{\mathbf{A}}_x = \left(\dot{A}^1_x, \dot{A}^2_x, \dot{A}^3_x\right) \tag{21.8}$$

$$\mathbf{E}_y = \dot{\mathbf{A}}_y = \left(\dot{A}^1_y, \dot{A}^2_y, \dot{A}^3_y\right) \tag{21.9}$$

$$\mathbf{B} = \frac{\partial \mathbf{A}_y}{\partial x} - \frac{\partial \mathbf{A}_x}{\partial y} - \mathbf{A}_x \times \mathbf{A}_y. \tag{21.10}$$

The bold-face symbols now represent vectors in a three dimensional color-space with superscript components labeled 1, 2 and 3. The cross product in the magnetic-field-analog is to be interpreted as a cross-product in this color-space. Here is the Lagrangian:

$$\mathcal{L} = \int dx\, dy \left(\mathbf{E}_x \cdot \mathbf{E}_x + \mathbf{E}_y \cdot \mathbf{E}_y - \mathbf{B} \cdot \mathbf{B}\right). \tag{21.11}$$

Apart from the cross product term in $\mathbf{B}$, this would just be three independent copies of the ordinary Maxwell electric and magnetic fields (one for each color). The new cross product term is also significant in contributing terms that are cubic and quartic in the vector potential. A Lagrangian with only quadratic terms, such as the Maxwell Lagrangian, produces linear equations and non-chaotic motion.

As before, we specialize to spatially uniform initial conditions so that the fields will be spatially uniform at all times. Here are the corresponding equations of motion:

$$\ddot{\mathbf{A}}_x = 2(\mathbf{A}_x \cdot \mathbf{A}_y)\mathbf{A}_y - 2(\mathbf{A}_y \cdot \mathbf{A}_y)\mathbf{A}_x \tag{21.12}$$

$$\ddot{\mathbf{A}}_y = 2(\mathbf{A}_x \cdot \mathbf{A}_y)\mathbf{A}_x - 2(\mathbf{A}_x \cdot \mathbf{A}_x)\mathbf{A}_y. \tag{21.13}$$

This is a system with six degrees of freedom. If the non-linear terms on the right hand side were absent, the most general solution — as in the Maxwell theory — would have constant $\mathbf{E}_x$ and $\mathbf{E}_y$, now with arbitrary directions in color-space.

The easiest way to convince you that the above equations for six degrees of freedom produce chaotic motion is to show that this behavior is displayed even by motion with simple initial conditions. It is easy to see that there are initial conditions such that the two vector potentials will always be of purely one color:

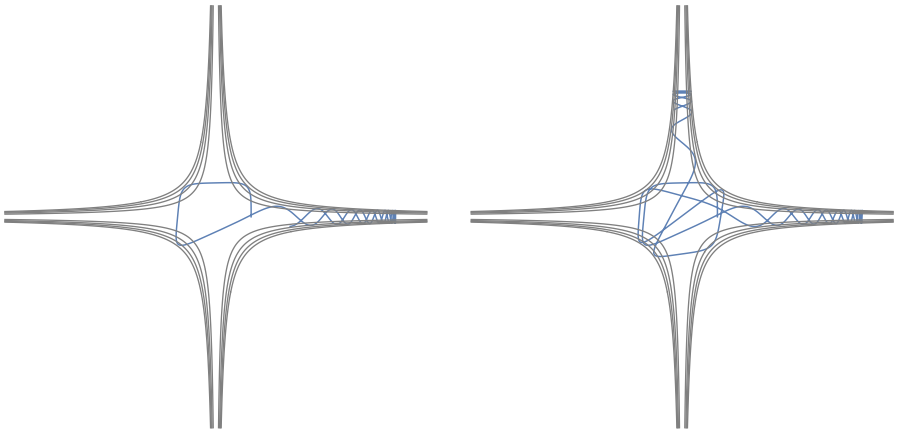
$$\mathbf{A}_x = (x(t),\, 0,\, 0) \tag{21.14}$$

$$\mathbf{A}_y = (0,\, y(t),\, 0). \tag{21.15}$$

The symbols $x(t)$ and $y(t)$ do not refer to position in space but the amplitudes of, respectively, the 1 and 2 components of color of the two vector potentials. By (21.12) and (21.13), the equations of motion are

$$\ddot{x} = -2y^2 x, \qquad \ddot{y} = -2x^2 y, \tag{21.16}$$

and correspond to a particle moving in the plane with a potential energy function $V = x^2 y^2$.



There is probably no better example of a mixing mechanism than the system of two degrees of freedom described by equations (21.16). The "particle" negotiates a potential, $V = x^2y^2$, with four infinite canyons running north, east, south and west from the origin. The potential has its minimum long the x and y axes and rises quadratically away from the axes, the curvature growing with distance from the origin. The more precisely the particle is launched along an axis, say the east canyon, the further it is able to move away from the origin. However, slight deviations from the axis are amplified by the growing curvature, resulting in higher and higher frequency oscillations perpendicular to the axis. Eventually the eastward progress of the particle is halted and its direction along the canyon is reversed. When it emerges back at the origin, after having experienced many oscillations in the canyon, it is just as likely to be injected into the north canyon as the south canyon (the west canyon is another possibility). The sequence of canyons that the particle enters is therefore very much like a random sequence.

The chaotic character of the dynamics of the Yang-Mills electric and magnetic fields that bind quarks no doubt plays a role in the structure of the proton. Unfortunately, to understand the proton one not only has to contend with classical motion that is inherently chaotic, but also the fact that the amount of action S involved is so small in units of $\hbar$ that it is necessary to use quantum mechanics.