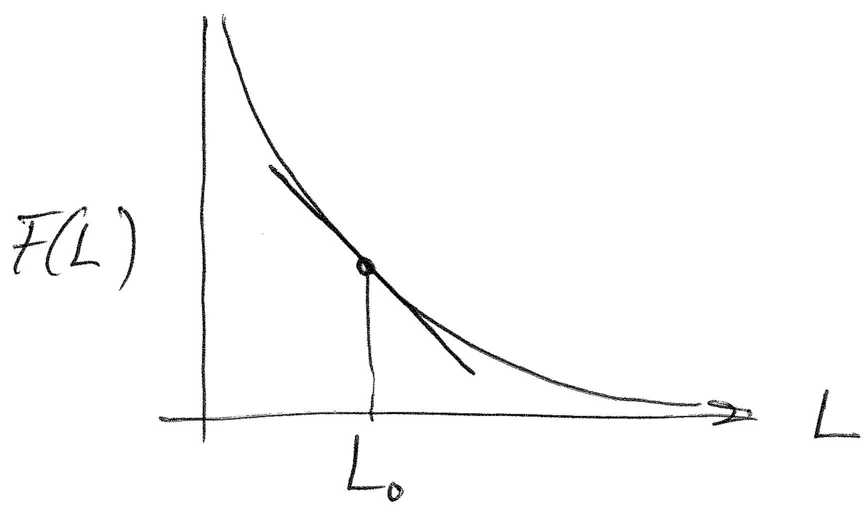
Lectures 21–22

force law from $E(L)$:

$$F(L) = -\frac{dE}{dL} = \frac{E(L_0)L_0}{L^2}$$

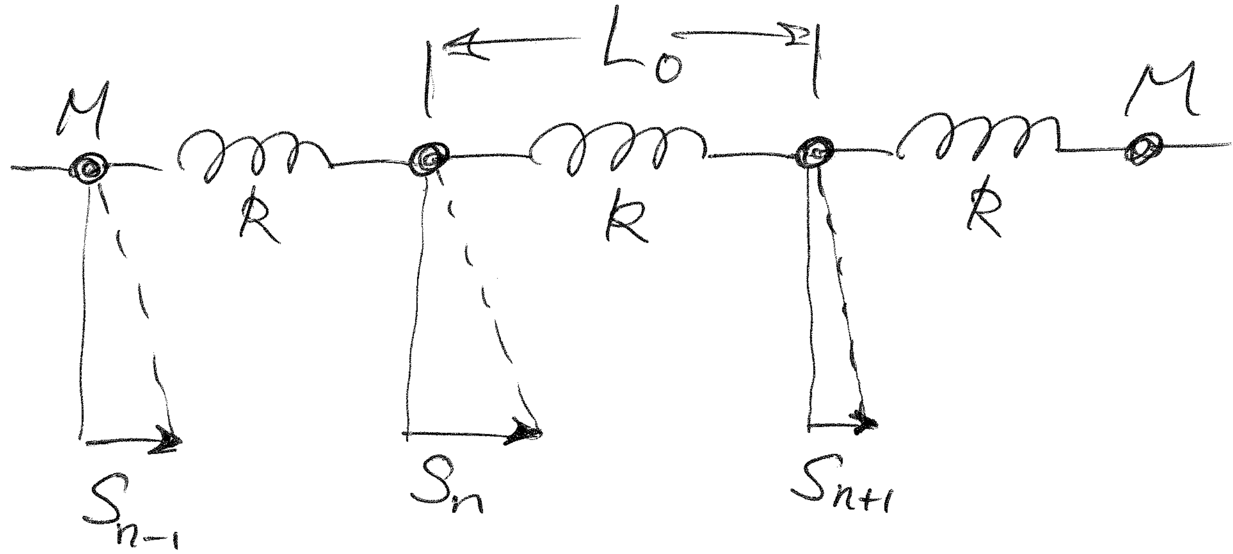
we linearize this about $L = L_0$ so we can model as Hookean spring:



$\Delta L = L - L_0$, $F \simeq F(L_0) - k\Delta L$

$$k = -\left.\frac{dF}{dL}\right|_{L_0} = 2\frac{E(L_0)}{L_0^2}$$

we can now reduce our system of partitioned gas volumes to mass points connected by springs — exactly as we did for the slinky:



$M = Nm$ (total mass between partitions)

$$k = 2\,\frac{E(L_0)}{L_0^2}$$

$$M\ddot{S}_n = F(L_0) - F(L_0) + k\left(S_{n+1} - S_n - (S_n - S_{n-1})\right)$$

22.3

$$M \ddot{S}_n = 2E(L_0)\underbrace{\left(\frac{S_{n+1} - 2S_n + S_{n-1}}{L_0^2}\right)}_{\frac{\partial^2 S}{\partial x^2}}$$

$$M \ddot{S}_n \rightarrow \frac{\partial^2 S}{\partial t^2}$$

$$v_s = \sqrt{\frac{2E(L_0)}{M}} = \text{sound speed}$$

Can relate this to actual <u>particle</u> speeds:

$$E(L_0) = N\left\langle \tfrac{1}{2} m v_p^2 \right\rangle$$

$$= \tfrac{1}{2} M \langle v_p^2 \rangle$$

$$\Longrightarrow \quad v_s = \underbrace{\sqrt{\langle v_p^2 \rangle}}_{\text{also called r.m.s. speed "root-mean-square"}}$$

The last equation is special to point particles in 2D. In a 3D gas the sound speed and r.m.s. particle speed are related by a number (which you know how to calculate!)

For sound in the 3D gas we will use a more general framework that uses the "bulk modulus" of the medium in which the sound propagates, B. We will calculate B for a 3D gas of point particles. However, our formula for sound speed v_s, which uses B, applies to a

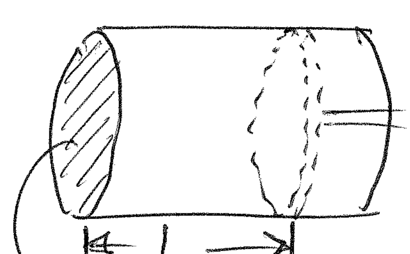
general medium where B (22.5)
might not be as easy to calculate
as it is for a gas!

Start with the energy law for
a cylinder of 3D gas (HW):

$$E(L) \propto 1/L^{2/3}$$

Now find the force law:

$$F(L) = -\frac{dE}{dL} \propto 1/L^{5/3}$$



area A

volume = $V = L \cdot A$

pressure = $p = F/A$

Note: A is just a
<u>constant</u> in all this

rewrite force law in terms of p and V:

$$p(V) \propto 1/V^{5/3}$$

Note that $p(V) \cdot V \neq$ const but $p(V)\,V^{5/3} =$ const., so we have something <u>different</u> from Boyle's law

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Now rewrite spring constant in terms of p and V:

$$k = -\left.\frac{dF}{dL}\right|_{L_0} = -A^2 \left.\frac{d(F/A)}{d(LA)}\right|_{L_0}$$

$$= -A^2 \left.\frac{dP}{dV}\right|_{V_0}$$

(22.7)

We could evaluate the derivative for the 3D gas, but let's keep it general so it applies to an arbitrary $P(V)$ function.

Use this k in our "slinky equation":

$$M \frac{\partial^2 S}{\partial t^2} = \left(-A^2 \frac{dP}{dV}\bigg|_{V_0}\right)\left(L_0^2 \frac{\partial^2 S}{\partial x^2}\right)$$

$$\underbrace{L_0^2 \frac{\partial^2 S}{\partial x^2}}_{S_{n+1} - 2S_n + S_{n-1}}$$

Note: $A^2 L_0^2 = V_0^2$

Define the bulk modulus:

$$B = -V \frac{dP}{dV}\bigg|_{V_0}$$

mass density:

$$\rho = \frac{M}{V_0}$$

These definitions will let us express the sound speed in a compact way. We can read off v_s^2 from the slinky equation:

$$v_s^2 = \frac{-V_0^2 \left.\frac{dP}{dV}\right|_{V_0}}{M} = \frac{B}{\rho}$$

This is very general — it applies even to liquids. In a liquid it's too complicated to calculate $E(V)$ (and its derivatives) and we have to <u>measure</u> B in an experiment.

Let's evaluate B for a 3D gas of point particles:

$$P \propto \frac{1}{V^\gamma} \quad (\gamma = 5/3)$$

standard notation for this exponent

$$\Rightarrow P(V) = P(V_0)\left(\frac{V_0}{V}\right)^\gamma$$

$$-V_0 \left.\frac{dP}{dV}\right|_{V_0} = -V_0\left(-\gamma\frac{P}{V_0}\right)\Bigg|_{V_0} = \gamma P(V_0)$$

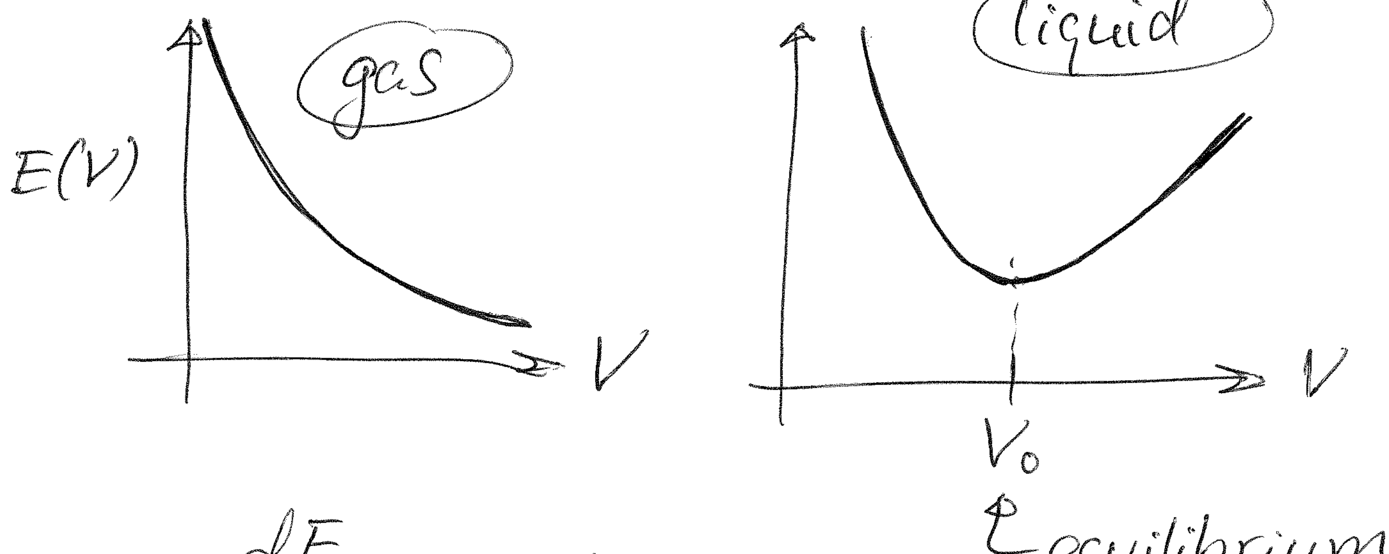
We see that for a gas B is proportional to P.

(22.10)

$$v_s = \sqrt{\gamma \frac{P}{\rho}}$$

(3D gas of point particles $\gamma = 5/3$)

Compare $E(V)$ for <u>gases</u> and <u>liquids</u>:



$P = -\frac{dE}{dV}$ is always positive for gases, but <u>negative</u> in a liquid if $V > V_0$

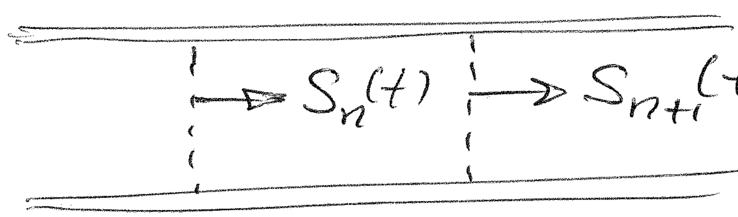
(a liquid expanded beyond equilibrium

volume is actually under <u>tension</u>)

$$B = -V\frac{dP}{dV} = +V\frac{d^2E}{dV^2} \quad \left(\begin{array}{c}\text{curvature}\\ \text{of } E(V)\\ \text{curve}\end{array}\right)$$

this is <u>positive</u> at all volumes for both gases and liquids.

When is "slow piston" approx. valid, which was used to derive $E(L)$.



we assumed $|V_x| = |\dot{S}| \ll v_p$

piston velocity

particle speed

Consider standing wave:

$$S(x,t) = A\cos(\omega t)\cos(kx)$$

$\omega = v_s k$.

A = max. "piston" displacement

$$\dot{s} = -A\omega \sin(\omega t) \cos(kx)$$

$$|\dot{s}|_{max} = A\omega \ll v_p$$

$$\Longrightarrow A(v_s k) \ll v_p$$

but we saw $v_s \simeq v_p$, so

$$A \ll 1/k = \lambda/2\pi$$

So "slow piston" approx. is good as long as the max. displacement (A) is a small fraction of the wavelength (λ)

There are two other things we need to check about our model. (1) We assumed the gas between partitions was <u>uniform</u> (in density, velocity distribution). This means L_0, the spacing of partitions, satisfies

$$L_0 \ll \lambda$$

because properties <u>do</u> change on the scale of the wavelength.

(2) We replaced the partitions by <u>pistons</u> and we want the change to the physics by this to be negligible. This will be true if the presence of the pistons is a "boundary effect". We want most of the collisions to be <u>between</u>

particles, and not between (22.14)
particles and pistons (which don't exist in the actual gas). For this to be the case, the mean-free-path l (distance between particle-particle collisions) must be much shorter than the distance between pistons:

$$l \ll L_0$$

Putting the last two conditions together,

$$l \ll L_0 \ll \lambda$$

we arrive at the following physical criterion:

$$l \ll \lambda$$

This sets a lower limit on

the sound wavelength:
for lengths shorter than this
our derivation cannot be
trusted.

Sound is usually described
as a pressure wave, and not
as wave motion of non-existent
partitions (although this is a
valid way to think about it).
Here is how we can relate the
two:

$$\left.\frac{dP}{dV}\right|_{V_0} = \gamma \frac{P_0}{V_0}$$

$$\Delta V = A(S_{n+1} - S_n) = AL_0\left(\frac{S_{n+1}-S_n}{L_0}\right)$$

$$= V_0 \frac{\partial s}{\partial x}$$

$$\Delta P = \frac{dP}{dV}\bigg|_{V_0} \cdot \Delta V = \gamma P_0 \frac{\partial s}{\partial x}$$

$$\Rightarrow \quad \Delta p \propto \frac{\partial s}{\partial x}$$

Using this relationship we can obtain a wave equation for Δp from the wave equation for s:

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 s}{\partial t^2}\right) = v_s^2 \frac{\partial}{\partial x}\left(\frac{\partial^2 s}{\partial x^2}\right)$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\left(\frac{\partial s}{\partial x}\right) = v_s^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial s}{\partial x}\right)$$

$$\Rightarrow \quad \frac{\partial^2 \Delta p}{\partial t^2} = v_s^2 \frac{\partial^2 \Delta p}{\partial x^2}$$