2.1 Additivity of angular velocity

As you learned in freshman physics, when the river has velocity $v_1$, and the kid swimming has velocity $v_2$ relative to the river, then the kid's velocity relative to the shore is $v_1 + v_2$. This is a simple consequence of the additivity of translations. If the position of a floating ball relative to the shore is $r_1$, and the position of the kid relative to the ball is $r_2$, then the position of the kid relative to the shore is $r_1 + r_2$. Take the time derivative of this and you get the addition of relative velocity rule.

Rotations are not that simple: they are not combined by addition. As a physical scenario, suppose there is a pendulum mounted on a merry-go-round. During time $\Delta t$ the pendulum (as body) has rotated about a horizontal axis relative to the merry-go-round (as “space”) by $U_1$. But the merry-go-round has been rotating during this time, so we have to take the result of the first rotation as the body-frame coordinates that get rotated by $U_2$, now relative to the fixed earth (true space). The net transformation of coordinates is therefore given by the (non-additive) product of rotations

$$U_{21} = U_2 U_1.$$ (2.1)

Fortunately, additivity still applies to angular velocity vectors. To see this, have the body and space frames coincide at time $t = 0$. We can do this because the body frame basis vectors are arbitrary, as long as we fix them once we've made our choice. As we learned in lecture 1, $\dot{U}_1 = A_1 U_1$, and therefore

$$U_1(\Delta t) \approx U_1(0) + \Delta t A_1(0) U_1(0)$$ (2.2)

is valid when $\Delta t$ is small. Since the two frames coincide at $t = 0$, $U_1(0) = 1$ and we have

$$U_1(\Delta t) \approx 1 + \Delta t A_1(0).$$ (2.3)

Here $A_1(0)$ is the antisymmetric matrix parametrized by the angular velocity vector $\omega_1$ of the pendulum relative to the merry-go-round at time $t = 0$. By exactly the same argument

$$U_2(\Delta t) \approx 1 + \Delta t A_2(0),$$ (2.4)

but where now $A_2(0)$ corresponds to the angular velocity vector $\omega_2$ of the merry-go-round relative to the earth. Taking the product

$$U_{21}(\Delta t) = U_2(\Delta t) U_1(\Delta t) \approx 1 + \Delta t (A_2(0) + A_1(0)) + O(\Delta t^2),$$ (2.5)

and comparing with the equation $\dot{U}_{21} = A_{21} U_{21}$, we see that

$$A_{21}(0) = A_2(0) + A_1(0).$$ (2.6)
Additivity of the antisymmetric matrix $A$ — the time-rate-of-change of $U$ — implies additivity of the associated angular velocity vectors:

$$\omega_{21} = \omega_2 + \omega_1.$$  \hfill (2.7)

Think of this as a statement about three frames, just like the kid swimming in the river. When frames 0 and 1 are related by angular velocity $\omega_1$, and frames 1 and 2 by angular velocity $\omega_2$, the upshot is that frames 0 and 2 are then related by their sum.

**Drawing exercise:** Make a sketch of the merry-go-round, pendulum, and $\omega_1$, $\omega_2$ and $\omega_{21}$ at a particular instant of time.

## 2.2 Fictitious forces

The body frame basis vectors are special cases of vectors fixed to the body whose time derivatives we worked out in lecture 1:

$$\dot{\hat{x}}' = \omega \times \hat{x}' \quad \dot{\hat{y}}' = \omega \times \hat{y}' \quad \dot{\hat{z}}' = \omega \times \hat{z}'. \hfill (2.8)$$

Now consider a general vector

$$\mathbf{v} = v_x' \hat{x}' + v_y' \hat{y}' + v_z' \hat{z}', \hfill (2.9)$$

where we allow the body frame components $v'_x(t)$, etc. to change with time. For example, if $\mathbf{v}$ were a position it would not be fixed in the body. Let’s compute the time derivative of this vector:

$$\dot{\mathbf{v}} = v'_x \dot{\hat{x}}' + v'_y \dot{\hat{y}}' + v'_z \dot{\hat{z}}' + v_x' \omega \times \hat{x}' + v_y' \omega \times \hat{y}' + v_z' \omega \times \hat{z}'$$

$$= \dot{\mathbf{v}} + \omega \times (v_x' \hat{x}' + v_y' \hat{y}' + v_z' \hat{z}')$$

$$= \dot{\mathbf{v}} + \omega \times \mathbf{v}. \hfill (2.10)$$

We’ll use an open circle above vectors to denote a frame-based time derivative:

$$\dot{\mathbf{v}} = \text{time derivative of } \mathbf{v} \text{ “as seen in the body frame”}.$$  

Equation (2.10) applies to *any* vector whose components we choose to express in terms of the rotating basis vectors $\hat{x}'$, $\hat{y}'$ and $\hat{z}'$. For example, when applied to $\mathbf{v} = \dot{\mathbf{r}}$ we get

$$\dot{\omega} = \dot{\omega}. \hfill (2.11)$$

The case we care most about is where our general vector $\mathbf{v}$ is the velocity vector

$$\mathbf{v} = \dot{\mathbf{r}} \hfill (2.12)$$

$$= \dot{\mathbf{r}} + \omega \times \mathbf{r}. \hfill (2.13)$$

Applying equation (2.10) to this vector we get

$$\dot{\mathbf{v}} = (\dot{\mathbf{r}} + \omega \times \dot{\mathbf{r}}) + \dot{\omega} \times \mathbf{r} + \omega \times (\dot{\mathbf{r}} + \omega \times \mathbf{r}) \hfill (2.14)$$

$$= \dot{\mathbf{r}} + \omega \times (\omega \times \mathbf{r}) + 2\omega \times \dot{\mathbf{r}} + \dot{\omega} \times \mathbf{r}. \hfill (2.15)$$
The point of the kinematical relationships above is to relate the true acceleration of a particle, $\ddot{v} = \ddot{r}$, to the apparent acceleration “as seen in the body frame”, $\ddot{r}$. Say the particle has mass $m$. The true force acting on the particle is

$$F_{\text{true}} = m\ddot{r},$$

(2.16)

while the force that “explains” the acceleration seen in the body frame is

$$F_{\text{body}} = m\dddot{r}.$$

(2.17)

Now if we insist on making sense of motion in the body frame — knowing full well that it is not an inertial frame — we can do so by introducing fictitious forces to make up the difference:

$$F_{\text{body}} = F_{\text{true}} + F_{\text{fict}},$$

(2.18)

$$F_{\text{fict}}/m = -\omega \times (\omega \times r) - 2\omega \times \dot{r} - \dot{\omega} \times r.$$  

(2.19)

The first two terms in the fictitious force have special names. The centrifugal force

$$F_{\text{cent}}/m = -\omega \times (\omega \times r)$$

(2.20)

scales as $\omega^2$ and depends on the position of the particle relative to the origin (axis of rotation). The Coriolis force

$$F_{\text{cor}}/m = -2\omega \times \dot{r}$$

(2.21)

scales as $\omega^1$ and applies only when the particle has a nonzero velocity ($\dot{r} \neq 0$) in the body frame. The third term in the fictitious force is zero or very small in many situations, such as Earth-bound observations, where the angular velocity vector is constant or nearly so.

**Question:** Explain the relationship between the power of $\omega$ and the number of time derivatives that is shared by all three fictitious forces.

**Question:** Consider the most commonly encountered situation, where $\omega$ is constant. One of the fictitious forces violates time-reversal symmetry — which one?