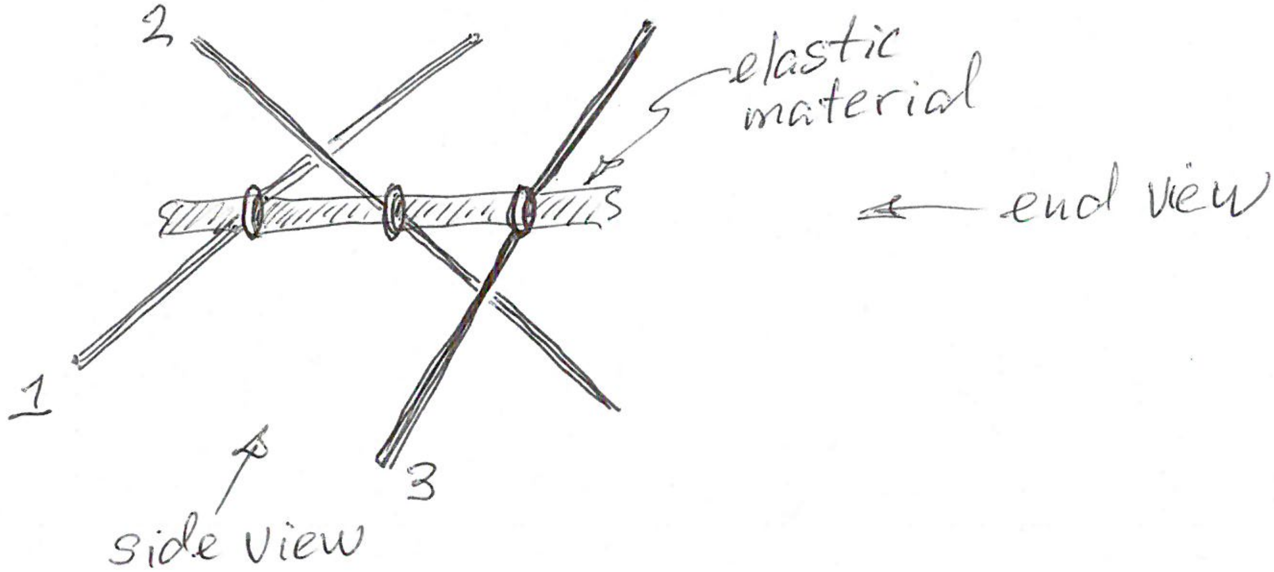
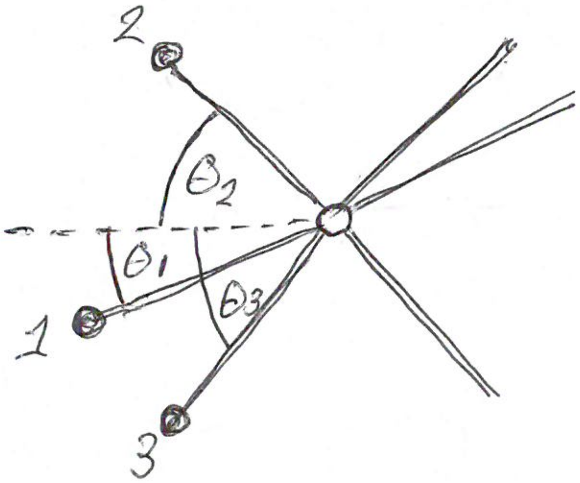
Lecture 1

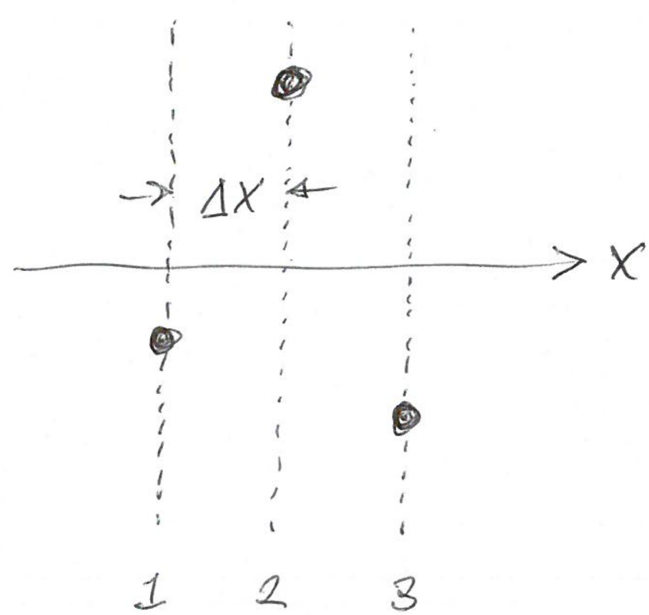
Torsional waves



end view



side view

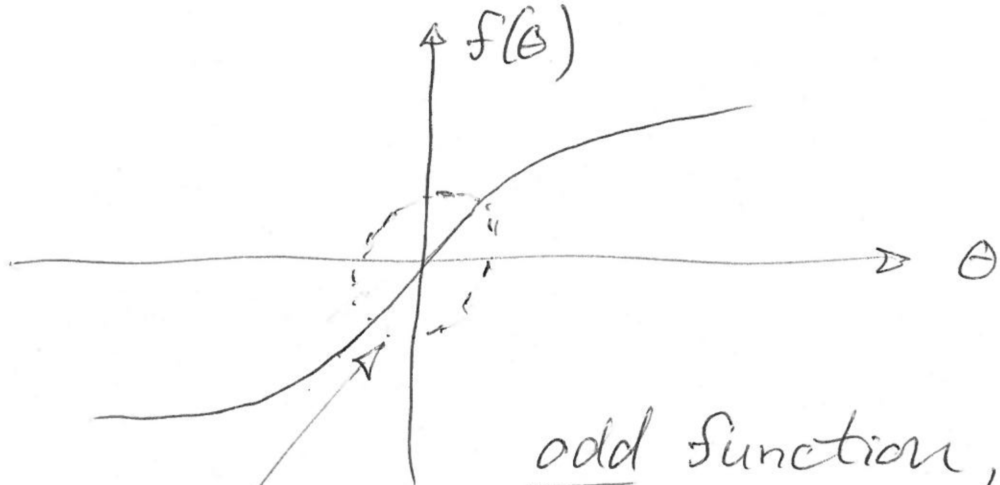


torque on 2 due to 1:

$$\tau_{21} = f(\theta_1 - \theta_2)$$

torque on 2 due to 3:

$$\tau_{23} = f(\theta_3 - \theta_2)$$



odd function, $f(0) = 0$

linearize f for small amplitudes:

$$f(\theta) \cong K\theta$$

net torque on 2:

$$\tau_2 = K(\theta_1 - \theta_2) + K(\theta_3 - \theta_2)$$

generalize:

$1 \rightarrow n-1$

$2 \rightarrow n$

$3 \rightarrow n+1$

Newton II :

$$I \ddot{\theta}_n = \tau_n = K(\theta_{n-1} - 2\theta_n + \theta_{n+1})$$

(system of ordinary differential equations)

I = moment of inertia of one rod

$$= \begin{cases} m\ell^2, & \text{all mass at end} \\ \frac{1}{3} m\ell^2, & \text{uniformly distributed mass} \end{cases}$$

ℓ = half-length of rod

Analysis strategy : Interpret RHS as derivative of $\theta(x,t)$ with respect to x ; $x = n \cdot \Delta x$

$$\frac{\partial \theta}{\partial x}(x,t) = \frac{\theta(x+\Delta x, t) - \theta(x,t)}{\Delta x}$$

$$= \frac{\theta_{n+1}(t) - \theta_n(t)}{\Delta x}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\Delta x}\left(\frac{\theta(x+\Delta x,t)-\theta(x,t)}{\Delta x} - \frac{\theta(x,t)-\theta(x-\Delta x,t)}{\Delta x}\right)$$

$$= \left(\frac{1}{\Delta x}\right)^2\left(\theta_{n+1}(t) - 2\theta_n(t) + \theta_{n-1}(t)\right)$$

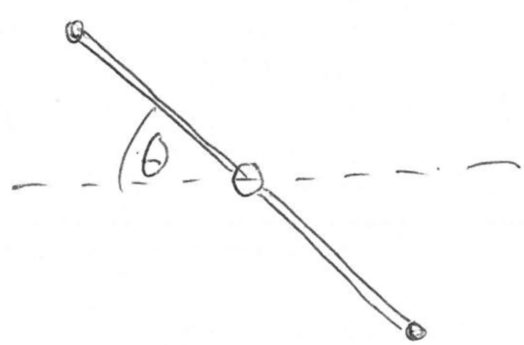
$$\ddot{\theta}_n = \frac{\partial^2\theta}{\partial t^2}(x,t)$$

Rewrite system of equations only in terms of partial derivatives:

$$I\,\frac{\partial^2\theta}{\partial t^2} = K(\Delta x)^2\,\frac{\partial^2\theta}{\partial x^2}$$

Reexpress constants:

single oscillator:



$$\tau = -K\theta$$

Newton II : $I\ddot{\theta} = -K\theta$

$$\ddot{\theta} = -\omega_0^2\theta \, , \quad \omega_0^2 = \frac{K}{I}$$

ω_0 = single-oscillator frequency

$$\frac{K(\Delta x)^2}{I} = (\omega_0 \Delta x)^2 = v^2$$

v = velocity = "wave velocity"

$$\frac{\partial^2\theta}{\partial t^2} = v^{-2}\frac{\partial^2\theta}{\partial x^2}$$ "wave equation"