Lecture 19

Let's apply the "loop rule" to the capacitor charging circuit:

The pure emf raises the potential by $+E$ as we go from $A$ to $B$. The voltage across the capacitor
is \( V = \frac{Q(t)}{C} \) (definition of \( \Theta \)) where higher potential terminal is the one with positive charge (since \( \vec{E} \) is directed from the positive plate to the negative plate). Therefore, if \( Q(t) > 0 \) in the diagram, the potential will drop by \( -\frac{Q(t)}{C} \) as we go from \( \Theta \) to \( \Theta' \). This change in potential is also correct if \( Q(t) < 0 \); in that case the potential increases (the direction of \( \vec{E} \) is reversed) and \( -\frac{Q(t)}{C} \) is a potential rise (positive quantity).
Finally, in going from $\text{(1)}$ to $\text{(3)}$ back to $\text{(2)}$ through the resistor, the potential drops by $-IR$ (if it turned out that $I$ was negative the potential in the resistor would drop when moving in the opposite direction — it always drops when moving in the direction of positive current). The three potential changes must bring us back to the original potential:

Loop rule: $+E - \frac{Q(t)}{C} - I(t)R = 0$

This equation contains two unknowns: $Q(t)$ and $I(t)$. We
need another equation that relates these unknowns. What's needed is the statement of charge conservation applied to the individual plates of the capacitor:

\[ \frac{dQ(t)}{dt} = I(t) \quad (\text{left plate}) \]

If this is satisfied then

\[ \frac{d(-Q(t))}{dt} = -I(t) \quad (\text{right plate}) \]

is also true and we have charge conservation on both plates.
Substituting \( \dot{Q} \) for \( I \) in the loop equation we get the following differential equation:

\[
E - \frac{Q}{C} - \dot{Q} R = 0
\]

The corresponding homogeneous equation \((E \to 0)\) is

\[
\dot{Q}_h = -\frac{1}{RC} Q_h
\]

with general solution

\[
Q_h(t) = Q_h(0) e^{-t/RC}
\]

A particular solution of the
loop equation is

\[ Q(t) = CE \ (= \ \text{const.}) \]

The most general solution is then this solution plus the most general solution of the homogeneous equation:

\[ Q(t) = CE + Q_h(0) e^{-t/RC} \]

Different choices of the constant \( Q_h(0) \) correspond to different scenarios or initial conditions.

If initially \( Q(t=0) \) the capacitor is uncharged, then we
want $Q_n(0) = -CE$.

charging solution:

$$Q(t) = CE(1 - e^{-t/\tau})$$

$\tau = RC = \text{"time constant"}$

Ohm $\times$ Farad = second

The charge approaches the static equilibrium ($I = 0$) value.
\[ Q = C E \] on an exponentially diminishing asymptote. A straight-line extrapolation of the initial slope,

\[ I(0) = \dot{Q}(0) = \frac{C E}{\tau} \]

would reach \( Q = C E \) in the time \( \tau \).

A "charged" or energized capacitor can be directly connected to a circuit without a battery and will drive a current. Our circuit diagram for charging
still applies if we set \( E=0 \) (no battery) and interpret the resistance as the device (light-bulb, etc.) we are driving. The equation is then

\[
\dot{Q} = -\frac{1}{\tau} Q
\]

with general solution:

\[
Q(t) = Q(0) e^{-t/\tau} \quad \text{"discharging"}
\]