Lecture 19: March 15

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19.1 Phase space

Phase space is the abstract space of which N dimensions correspond to the N generalized coordinates and another N their corresponding conjugate momenta. With twice as many dimensions as ordinary coordinate space, phase space can convey, geometrically, not just one trajectory of the system but the full range of motions that are possible. We will illustrate this "global" perspective on motion with the example of a pendulum that may swing with arbitrary amplitude, even to the point of looping around its support.

The arm of the pendulum, constrained to swing in a plane, has length l and negligible mass compared with the mass M attached to its end. The Lagrangian for this pendulum is

$$\mathcal{L} = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl \cos\theta, \tag{19.1}$$

where θ is the angle with respect to the vertical. From this we obtain the conjugate momentum

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2 \dot{\theta}, \tag{19.2}$$

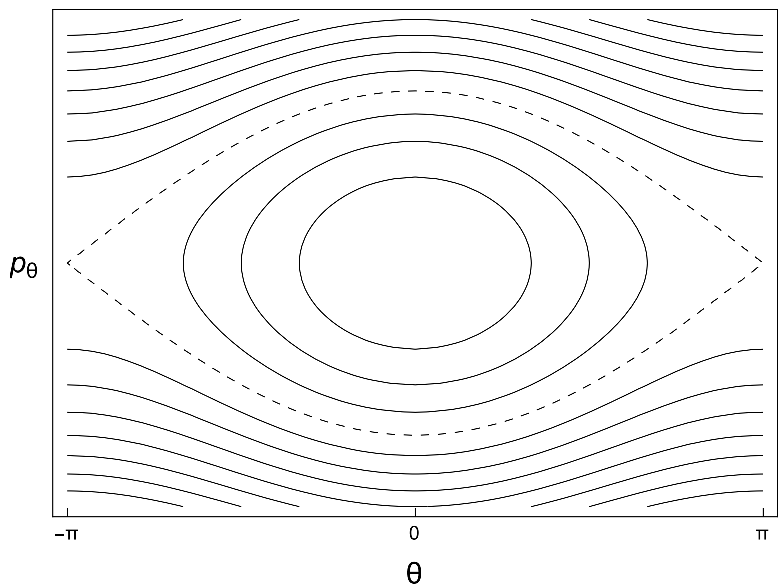
and the Hamiltonian

$$\mathcal{H} = \dot{\theta}\, p_\theta - \mathcal{L} = \frac{p_\theta^2}{2ml^2} - mgl \cos\theta. \tag{19.3}$$

Since $d\mathcal{H}/dt = -\partial\mathcal{L}/\partial t = 0$, trajectories stay on the "surface"

$$\mathcal{H}(\theta, p_\theta) = E = \text{constant}, \tag{19.4}$$

where "surface" means one dimension less than the dimension of phase space, or $2N - 1$. For our simple pendulum the energy surfaces are the one-dimensional contours shown in the figure on the next page. We recognize the angular character of the variable θ by picturing only one period in its range. While the phase space looks like it would be an infinite strip, it is actually (in a topological sense) the infinite cylinder formed by joining two sides where θ differs by 2π (in the figure we chose $\theta = \pm\pi$).



Part of the pendulum's two dimensional phase space showing contours of constant energy. The dashed contour separates three qualitatively different modes of motion. Since the variables on the left edge of this plot ($\theta = -\pi$) are equivalent to the variables on the right edge ($\theta = \pi$), the pendulum's phase space is topologically an infinitely long cylinder.

19.1.1 Hamiltonian flow

Hamilton's equations for the pendulum

$$\dot{\theta} = +\frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{ml^2} \tag{19.5}$$

$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = -mgl\sin\theta, \tag{19.6}$$

define the "flow" on the energy surfaces.

Exercise: Add arrows to the energy contours above that show how the pendulum moves in phase space.

The flow field defined by Hamilton's time evolution equations divide up the phase space of the pendulum into three distinct domains:

- Bounded oscillations with center at $p_\theta = 0$, $\theta = 0$.
- Clockwise spinning ($\dot{\theta} < 0$).
- Counterclockwise spinning ($\dot{\theta} > 0$).

Trajectories in phase space may never cross, otherwise there would be points where the flow vector $(\dot{\theta}, \dot{p}_\theta)$ is not uniquely determined by the value of $(\partial\mathcal{H}/\partial p_\theta, -\partial\mathcal{H}/\partial\theta)$. An exception might be made for points such as $(\theta = \pm\pi, p_\theta = 0)$, where the flow vector is zero. But these are better classified as unstable equilibrium points. Trajectories arbitrarily close to these points never actually cross.

Exercise: Make a pair of sketches contrasting the flows in the neighborhoods of stable and unstable equilibrium points.

Uniqueness of the flow also excludes the following from happening. Consider an arbitrary region $A(0)$ in phase space. For example, an $A(0)$ that is a small disk ($2N$-ball) centered at some point has the interpretation of many nearly identical initial conditions. Now consider the region $A(t)$ formed by time-evolving the points of $A(0)$ for time t with Hamilton's equations. Can the boundary of $A(t)$ impinge on itself — that is, can $A(t)$ self-intersect? The best way to see that this cannot happen is to consider the time-reversed Hamilton equations, for which the flow is no less unique (a point in the alleged region twice-covered by $A(t)$ would then flow to different points in the past).

Not only is the topology of an arbitrary region of phase space $A(t)$ preserved over time by the uniqueness of Hamiltonian flow, an even stronger property holds: its area ($2N$-volume) is constant. This is the subject of the next section.

19.1.2 Liouville's theorem

The constancy of the volume of a time-evolving region in phase is known as *Liouville's theorem.* A statement of the theorem that follows directly from Hamilton's equations is

> **Liouville's theorem 1:** The Hamiltonian flow field has zero divergence.

The Hamiltonian flow field for a system with N degrees of freedom has components

$$(\dot{q}_1, \ldots, \dot{q}_N\,;\, \dot{p}_1, \ldots, \dot{p}_N). \tag{19.7}$$

In particular, the q_k component of the field has value

$$\dot{q}_k = \frac{\partial \mathcal{H}}{\partial p_k}. \tag{19.8}$$

Similarly, the p_k component of the field has value

$$\dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q_k}. \tag{19.9}$$

The divergence (in $2N$ dimensions) is obtained by taking the q_k-partial derivative of the q_k component, the p_k-partial derivative of the p_k component, and summing these together and also over all k from 1 to N. The resulting divergence is

$$\sum_{k=1}^{N} \frac{\partial}{\partial q_k}\left(\frac{\partial \mathcal{H}}{\partial p_k}\right) + \sum_{k=1}^{N} \frac{\partial}{\partial p_k}\left(-\frac{\partial \mathcal{H}}{\partial q_k}\right) = 0. \tag{19.10}$$

To understand the connection between a flow field with vanishing divergence and volume-constancy, we consider a general model of flow:

$$\dot{x}_i = F_i(x_1, \ldots, x_M), \qquad i = 1, \ldots, M. \tag{19.11}$$

Here x is a point in an M-dimensional space. Our Hamiltonian flow problem corresponds to $M = 2N$ and a particular form for the functions F_i. By time-evolving a short time Δt we can define a coordinate transformation

$$x_i \to x'_i(x) = x_i + \Delta t\, F_i(x). \tag{19.12}$$

The Jacobian J of this coordinate transformation tells us how the volumes of integration elements are transformed. Evaluating the Jacobian at the point $x = a$ we obtain

$$J(a) = \det G(a), \qquad G_{ij}(a) = \left.\frac{\partial x'_i}{\partial x_j}\right|_a = \delta_{ij} + \Delta t \left.\frac{\partial F_i}{\partial x_j}\right|_a . \tag{19.13}$$

Since the $M \times M$ matrix $G(a)$ has the form

$$G(a) = \mathbb{1} + \Delta t\, F', \qquad F'_{ij} = \left.\frac{\partial F_i}{\partial x_j}\right|_a , \tag{19.14}$$

we can expand its determinant in powers of Δt:

$$J(a) = \det G(a) = 1 + \Delta t\, \mathrm{Tr}(F') + O(\Delta t^2) = 1 + \Delta t \sum_{i=1}^{M} \left.\frac{\partial F_i}{\partial x_i}\right|_a + O(\Delta t^2). \tag{19.15}$$

We see that there is no volume change to order Δt whenever the divergence of the flow field vanishes[1]. Because the Hamiltonian flow field is a flow field with vanishing divergence, we have the second form of Liouville's theorem:

> **Liouville's theorem 2:** The phase space volume of a region $A(t)$ evolved over time t by Hamiltonian flow is constant.

[1]The higher order terms in Δt do not need to be considered since the transformation (19.12) is only specified to lowest order.