Lectures 19-20

Sound in a Gas

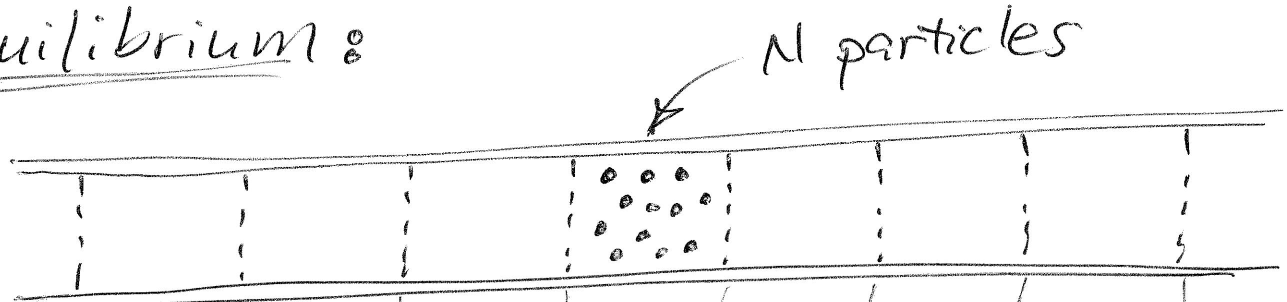
our approach: basic mechanics + statistical hypotheses

---

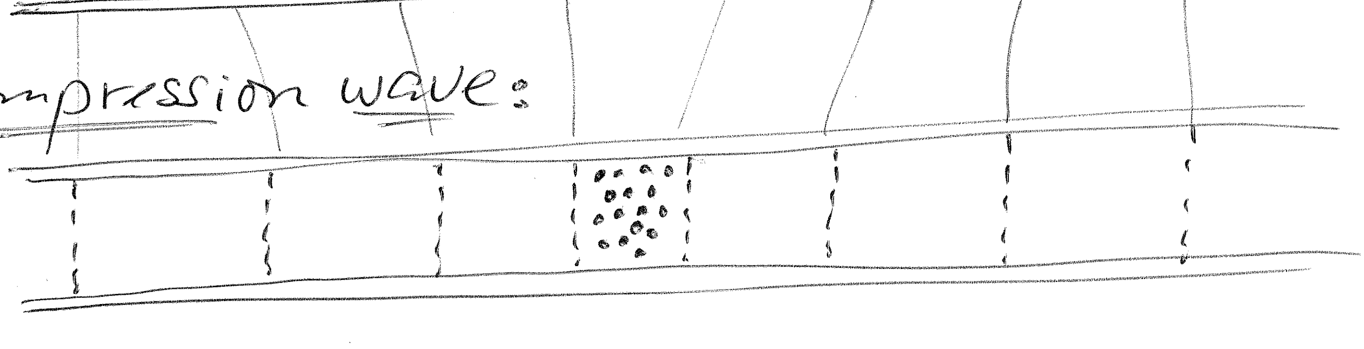
start with 2D "pipe" (graduate to 3D later)

partition gas into equal parcels of N particles (atoms, molecules, ...)

equilibrium:

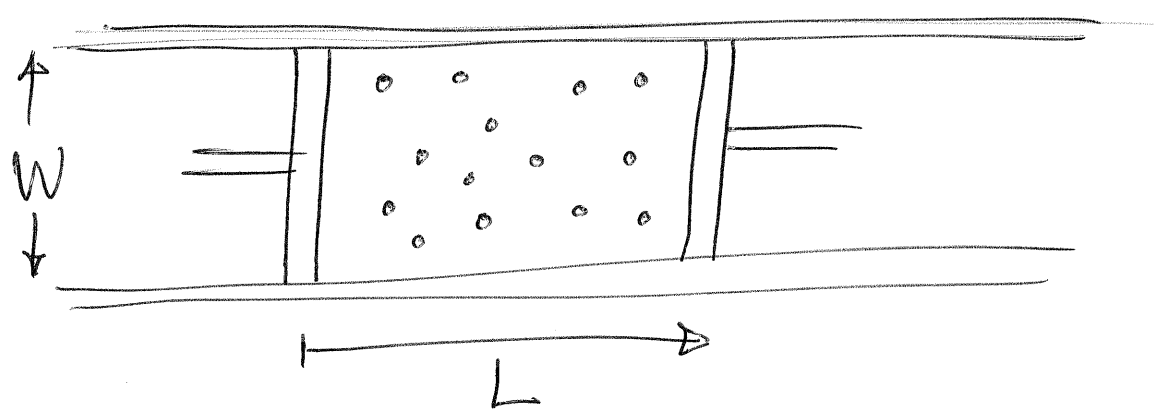


compression wave:



model "elastic" properties of one gas parcel and represent as a spring for wave analysis

place moveable pistons at ends:

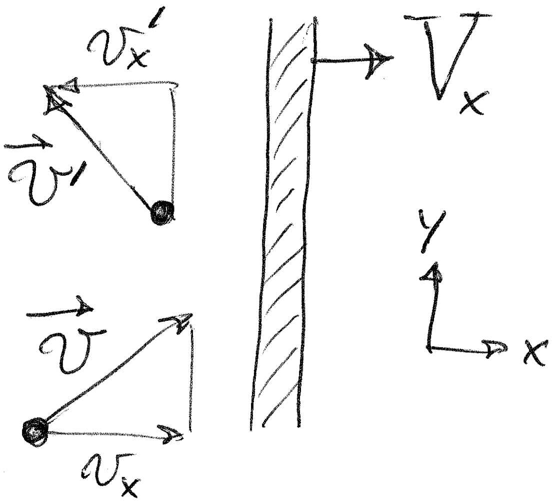


$E(L)$ = kinetic energy of gas particles as function of "cylinder" length

plan: calculate change in $E(L)$ due to moving piston and use work-energy theorem of mechanics

20.3

change in energy of **one** gas particle upon collision with moving piston:



$\vec{v}$ = velocity before

$\vec{v}'$ = velocity after

assumptions: $|V_x| \ll v_x$,

$v_x > 0$.

since no collision if $v_x < 0$

$$v_x' = -(v_x - V_x) + V_x = 2V_x - v_x$$

$$v_y' = v_y$$

moving frame calc. from HW

change in Kinetic energy:

$$\Delta E_1 = \tfrac{1}{2}m(v_x'^2 + v_y'^2) - \tfrac{1}{2}m(v_x^2 + v_y^2)$$

one particle

$$= \tfrac{1}{2}m\left((2V_x - v_x)^2 - v_x^2\right)$$

$$\simeq -2mV_x v_x + \cancel{2mV_x^2}$$

neglect: $|V_x| \ll v_x$

---

to deal with all particles, in an average sense, we need to know <u>distributions</u> of positions and velocities.

We take as hypotheses the distributions that have the maximum symmetry (i.e. uniformity) as we expect of a gas "in equilibrium" we expect these hypotheses will still be valid when the confines (i.e. pistons) of the gas are changed very slowly ($|V_x| \ll v_x$)

statistical hypotheses

- distribution of positions remains uniform

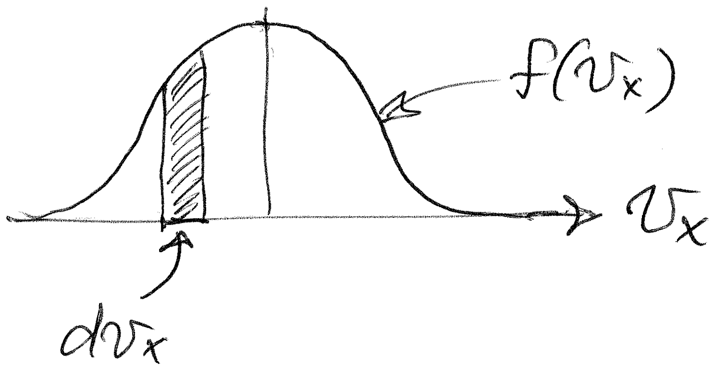
- distribution of velocities remains isotropic (uniform in direction) and independent of position.

So where ever we look in a slowly expanding cylinder of gas there will be the same density

(20.6)

of particles and their velocities will have the same, isotropic distribution.

$f(v_x)\,dv_x$ = fraction of particles whose x-velocity is in range $(v_x, v_x + dv_x)$ (anywhere in the gas)



isotropy: $f(v_x) = f(-v_x)$

averages: $\langle v_x \rangle = \int_{-\infty}^{\infty} v_x\, f(v_x)\, dv_x$

$= 0$

20.7

$$\langle v_x^2 \rangle = \int_{-\infty}^{\infty} v_x^2 f(v_x)\, dv_x \neq 0$$

will need this later:

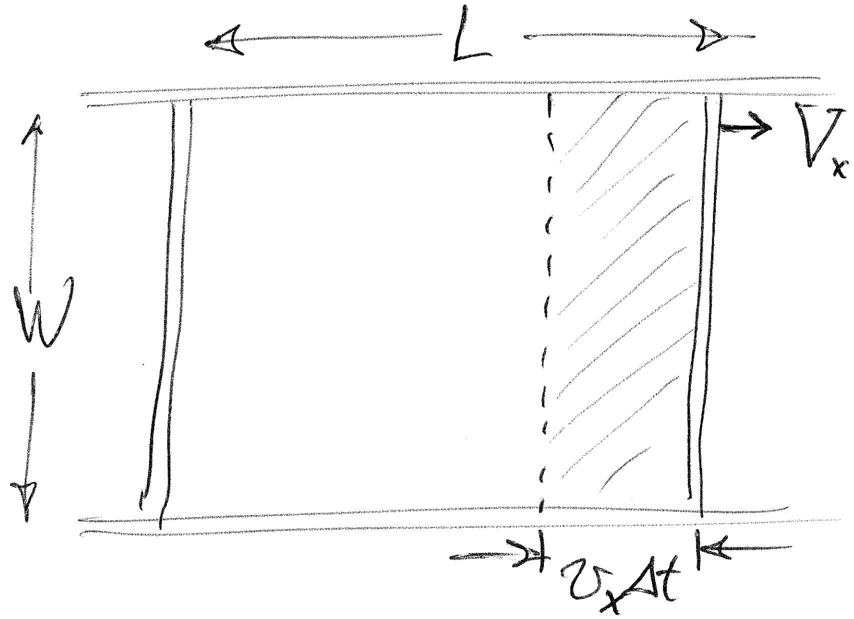
$$\int_0^{\infty} v_x^2 f(v_x)\, dv_x = \frac{1}{2} \int_{-\infty}^{\infty} v_x^2 f(v_x)\, dv_x$$

zero

$$= \frac{1}{2} \langle v_x^2 \rangle$$

---

ready to calculate change in energy of all particles during time Δt:



(20.8)

$$\begin{pmatrix} \text{number of particles with} \\ v_x \text{ in range } v_x + dv_x \text{ that} \\ \text{collide during time } \Delta t \end{pmatrix} =$$

$$= N \underbrace{\left(\frac{v_x \Delta t}{L}\right)}_{\text{area fraction}} \underbrace{f(v_x)\, dv_x}_{\text{fraction with correct } v_x}$$

$$\begin{pmatrix} \text{energy change of particles} \ldots \\ \text{(same as above)} \end{pmatrix}$$

$$= N \left(\frac{v_x \Delta t}{L}\right) f(v_x)\, dv_x \left(-2m V_x v_x\right)$$

(20.4)

$$\begin{pmatrix} \text{energy change of all particles} \\ \text{that collide in time } \Delta t \end{pmatrix}$$

$$= \Delta E = \int_0^\infty \left(-2mV_x N \frac{\Delta t}{L}\right) v_x^2 f(v_x)\, dv_x$$

no collisions for negative v_x

$$V_x \Delta t = \Delta L$$

$$\Delta E = -2mN \frac{\Delta L}{L} \frac{1}{2} \langle v_x^2 \rangle$$

use isotropy of velocity again:

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle$$

relate to total kinetic energy E:

$$E = N \left\langle \tfrac{1}{2} m (v_x^2 + v_y^2) \right\rangle$$

$$= N \cdot \tfrac{1}{2} m \left(\langle v_x^2 \rangle + \langle v_y^2 \rangle \right)$$

$$= N \tfrac{1}{2} m \left(2 \langle v_x^2 \rangle \right) \quad \text{(isotropy)}$$

$$E = N m \langle v_x^2 \rangle$$

---

use in earlier result:

$$\Delta E = -\frac{\Delta L}{L} E$$

$$\frac{\Delta E}{E} = -\frac{\Delta L}{L}$$

Solve this by integration (20.11)

$$\frac{dE}{E} = - \frac{dL}{L}$$

$$\int_{E_0}^{E_1} \frac{dE}{E} = - \int_{L_0}^{L_1} \frac{dL}{L}$$

$$\log\left(E_1/E_0\right) = - \log\left(L_1/L_0\right)$$

$$E_1 = E_0\left(\frac{L_0}{L_1}\right)$$

$$E_0 = E(L_0) \, , \quad E_1 = E(L_1)$$

$$E(L_1) = E(L_0)\left(\frac{L_0}{L_1}\right) \propto \frac{1}{L_1}$$