Lecture 17

make

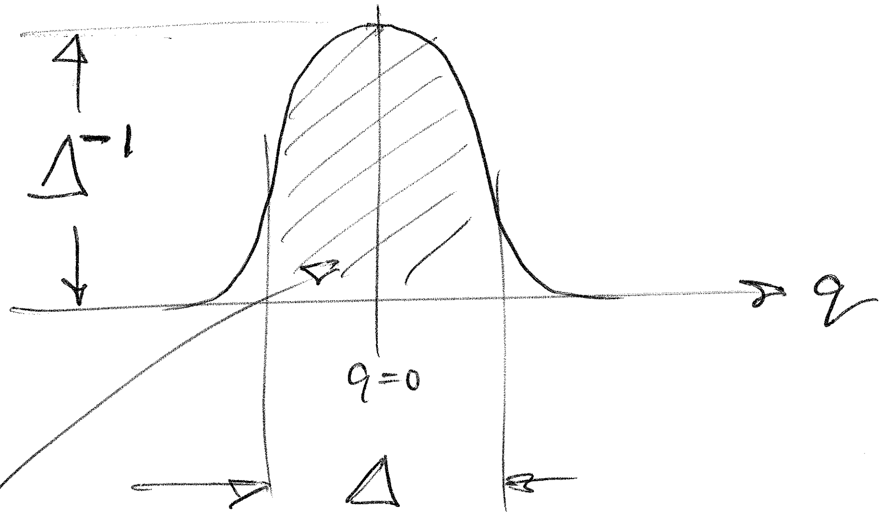
$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{i(k'-k)x}$$

well-defined with a "convergence-factor":

$$f(q) \equiv \lim_{\Delta \to 0} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{i(k'-k)x - \Delta^2 x^2}}_{\text{converges for arbitrarily small } \Delta}$$

integral is essentially of the same form as integral in lecture 14 (completing-the-square, etc.)

$$f(q) = \lim_{\Delta \to 0} \frac{e^{-\frac{q^2}{4\Delta^2}}}{\sqrt{4\pi\Delta^2}}$$



Area constant as $\Delta \to 0$

$$\int f(q)\,dq = 1$$

$f(q) =$ "Dirac delta function"

$= \delta(q)$

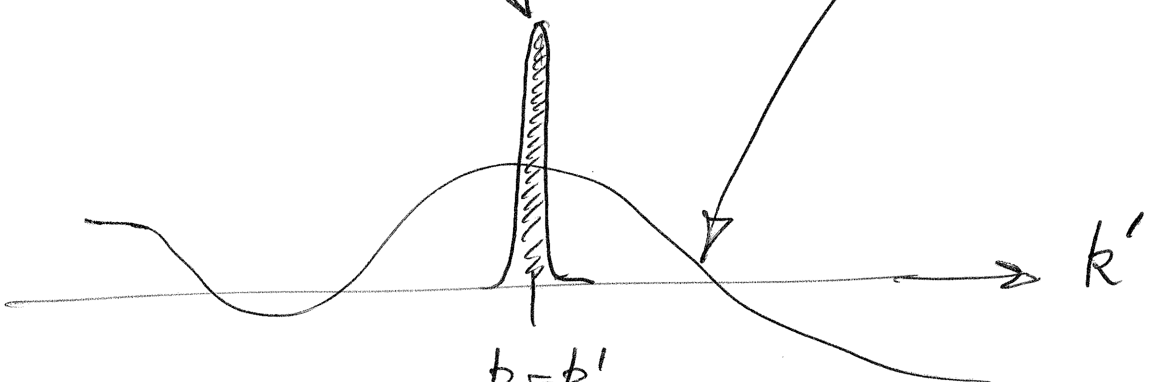
properties:

$\delta(q) = 0, \quad q \neq 0$

$\int \delta(q)\,dq = 1$

Return to integral of lecture 16:

$$\hat{S}(k,t) \stackrel{?}{=} \int_{-\infty}^{\infty} dk' \; \delta(k'-k)\, \hat{S}(k',t)$$



$$\delta(k'-k)\,\hat{S}(k',t) = \delta(k'-k)\,\hat{S}(k,t)$$

$$\left[\begin{array}{l} \text{either } 0=0 \quad (k \neq k') \\ \qquad\qquad \text{or} \\ \hat{S}(k',t) = \hat{S}(k,t) \quad (k=k') \end{array}\right]$$

$$\hat{S}(k,t) \stackrel{?}{=} \int_{-\infty}^{\infty} dk' \, \delta(k'-k)\, \hat{S}(k,t)$$

(constant)

$$\hat{S}(k,t) \stackrel{?}{=} \hat{S}(k,t) \underbrace{\int_{-\infty}^{\infty} \delta(k'-k)\, dk'}_{1} \qquad (17.4)$$

we have verified "inverse-transform" formula

<u>Most</u> <u>general</u> right-moving wave:

$$S_R(x,t) = \int_0^{\infty} h(k) \cos(kx - \omega(k)t + \phi(k))\, dk$$

same as we used when constructing wavepackets, except $h(k)$ need <u>not</u> be narrow distribution about some k_0, and $\phi(k)$ allows for relative phase lags of sinusoids with different k.

Rewrite $S_R(x,t)$ to make it look like a Fourier transform:

$$S_R(x,t) = \int_0^\infty dk \left[\tfrac{1}{2} h(k) e^{i\phi(k) - i\omega(k)t + ikx} + \tfrac{1}{2} h(k) e^{-i\phi(k) + i\omega(k)t - ikx} \right]$$

interpret second term as due to an integral from $-\infty$ to 0:

$$S_R(x,t) = \int_0^\infty dk \left[F(k) e^{ikx} + F(-k) e^{-ikx} \right]$$

(for appropriately defined $F(k)$ - see next page)

then $S_R(x,t) = \int_{-\infty}^{\infty} dk\, F(k) e^{+ikx}$ (*)

$k > 0$:

$F(k) = $ (first term)

$= \frac{1}{2} h(k) e^{i\phi(k) - i\omega(k)t}$

$k < 0$:

$F(k) = $ (second term)

$= \frac{1}{2} h(-k) e^{-i\phi(-k) + i\omega(-k)t}$

Note that equation (*) is exactly how we defined Fourier transforms. Hence:

$\hat{S}(k,t) = F(k)$ (as defined above)