

## Lecture 17: March 10

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## 17.1 Conservation laws from continuous symmetries

No doubt you have come across the assertion that the symmetries of space are somehow responsible for the fundamental conservation laws of linear and angular momentum. In this lecture we see how this connection is made precise. The key idea is to study the properties of a Lagrangian that is invariant with respect to a *continuous transformation*. A theorem discovered (and proved) by Emmy Noether in 1915 provides a method for teasing out the conserved quantity directly from the Lagrangian, provided there is an invariance with respect to a transformation law. Beyond its obvious intellectual appeal, the theorem is useful in exotic spheres of physics (e.g. elementary particle theory) where an intuitive/geometric understanding of symmetries can be elusive!

### 17.1.1 Continuous symmetries of the Lagrangian

Let's start with an example. Consider the Lagrangian for a particle freely moving in the plane (no potential energy). In polar coordinates:

$$\mathcal{L} = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2). \quad (17.1)$$

As we learned in lecture 10, the absence of  $\theta$  in  $\mathcal{L}$  implies that the momentum conjugate to  $\theta$ ,

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = Mr^2\dot{\theta} \quad (17.2)$$

is conserved — the angular momentum. Surprisingly, this system has another conservation law associated with a mysterious continuous symmetry known as “*s*-symmetry”.

The transformation rule for a continuous symmetry only needs to be specified close to the identity transformation. “Large” transformations are then obtained by composing the infinitesimal transformations. We will use the variable  $s$  to parameterize the transformation of *s*-symmetry, where  $s = 0$  corresponds to the identity transformation. The near-identity (lowest order in  $s$ ) transformation rule is written as follows:

$$r(s) = r + (\cos \theta)s \quad (17.3)$$

$$\theta(s) = \theta - \left(\frac{\sin \theta}{r}\right)s. \quad (17.4)$$

To study the transformation of the Lagrangian (17.1) we also need to work out the transformation of the

velocities:

$$\dot{r}(s) = \dot{r} - (\dot{\theta} \sin \theta) s \quad (17.5)$$

$$\dot{\theta}(s) = \dot{\theta} - \left( \frac{\dot{\theta} \cos \theta}{r} \right) s + \left( \frac{\dot{r} \sin \theta}{r^2} \right) s. \quad (17.6)$$

Substituting these and the transformation  $r(s)$  into (17.1),

$$\mathcal{L}(s) = \frac{1}{2} M \left( \left( \dot{r} - (\dot{\theta} \sin \theta) s \right)^2 + \left( r + (\cos \theta) s \right)^2 \left( \dot{\theta} - \left( \frac{\dot{\theta} \cos \theta}{r} \right) s + \left( \frac{\dot{r} \sin \theta}{r^2} \right) s \right)^2 \right), \quad (17.7)$$

expanding and keeping terms only up to linear order in  $s$ , we find:

$$\mathcal{L}(s) = \frac{1}{2} M (\dot{r}^2 + r^2 \dot{\theta}^2) + O(s^2). \quad (17.8)$$

This shows that the Lagrangian is unchanged to lowest order in the parameter  $s$ . It should not bother you that there are terms of order  $s^2$ , because we only went as far as order  $s^1$  in expressing the transformation rule (17.3). Our formal definition of invariance is the property

$$\left. \frac{d}{ds} \mathcal{L}(s) \right|_{s=0} = 0. \quad (17.9)$$

### 17.1.2 Conservation laws from continuous symmetries

We now turn to the derivation of a conservation law for the general Lagrangian

$$\mathcal{L}(q_1(s), \dots, q_N(s); \dot{q}_1(s), \dots, \dot{q}_N(s)), \quad (17.10)$$

where  $s$  is the parameter of a continuous transformation, such as the mysterious  $s$ -symmetry described above. The derivation goes through when we have the invariance property

$$\left. \frac{d}{ds} \mathcal{L}(q_1(s), \dots, q_N(s); \dot{q}_1(s), \dots, \dot{q}_N(s)) \right|_{s=0} = 0. \quad (17.11)$$

The first step in the derivation is to apply the multi-variable chain rule in evaluating the derivative:

$$\sum_{k=1}^N \left( \frac{\partial \mathcal{L}}{\partial q_k} \frac{dq_k}{ds} + \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \frac{d\dot{q}_k}{ds} \right) \Big|_{s=0} = 0. \quad (17.12)$$

Next we use the fact that the Euler-Lagrange equations still hold (for any fixed  $s$ ), and therefore:

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right), \quad k = 1, \dots, N. \quad (17.13)$$

The result of substituting this into (17.12),

$$\sum_{k=1}^N \left( \left( \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) \right) \frac{dq_k}{ds} + \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \left( \frac{d}{dt} \left( \frac{dq_k}{ds} \right) \right) \right) \Big|_{s=0} = 0, \quad (17.14)$$

is the product rule of differentiation applied to

$$\frac{d}{dt} \left( \sum_{k=1}^N \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \frac{dq_k}{ds} \right) \Big|_{s=0} = 0. \quad (17.15)$$

Using the definition of conjugate momentum (at  $s = 0$ )

$$\left. \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right|_{s=0} = p_k, \quad (17.16)$$

we can write (17.15) as

$$\frac{dI}{dt} = 0, \quad I = \sum_{k=1}^N p_k \left. \frac{dq_k}{ds} \right|_{s=0}. \quad (17.17)$$

This shows that the quantity  $I$ , constructed from the conjugate momenta and the rates of change of the generalized coordinates with respect to  $s$ , is conserved.

Let's evaluate  $I$  for the mysterious  $s$ -symmetry of a particle moving in the plane:

$$p_r = M\dot{r} \quad \left. \frac{dr(s)}{ds} \right|_{s=0} = \cos \theta, \quad (17.18)$$

$$p_\theta = Mr^2\dot{\theta} \quad \left. \frac{d\theta(s)}{ds} \right|_{s=0} = -\frac{\sin \theta}{r}. \quad (17.19)$$

$$I = (M\dot{r})(\cos \theta) + (Mr^2\dot{\theta}) \left( -\frac{\sin \theta}{r} \right) \quad (17.20)$$

$$= M(\dot{r} \cos \theta - r\dot{\theta} \sin \theta). \quad (17.21)$$

**Question:** Is this really a “new” conserved quantity? Describe the mysterious  $s$ -symmetry in non-mysterious terms.

### 17.1.3 Noether's theorem

A slight generalization of the statement of invariance (17.11),

$$\left. \frac{d}{ds} \mathcal{L}(q_1(s), \dots, q_N(s); \dot{q}_1(s), \dots, \dot{q}_N(s)) \right|_{s=0} = \frac{dF}{dt}, \quad (17.22)$$

still leads to a conserved quantity. Here  $F$  is an arbitrary function of the generalized coordinates and their velocities. Repeating the derivation above with  $dF/dt$  replacing the zero on the right-hand side, we see that the conserved quantity is now

$$I = \sum_{k=1}^N p_k \left. \frac{dq_k}{ds} \right|_{s=0} - F. \quad (17.23)$$

This more general construction of a conserved quantity from a continuous symmetry is known as Noether's theorem. In a homework problem you will see an application where the function  $F$  appears.

**Question:** Does adding the time derivative of a function,  $dF/dt$ , to a Lagrangian (say as a result of a continuous symmetry transformation) change the equations of motion? *Hint:* How does adding such a term affect Hamilton's principle?