16.1 Constrained variations in mechanics (continued)

16.1.1 Non-holonomic constraints: the rolling wheel

Recall that the Lagrangian formulation of mechanics is incomplete unless we can adapt the method to handle systems with non-holonomic constraints. In such systems it is impossible to specify the positions of all the particles with \( N \) generalized coordinates, where \( N \) is the number of degrees of freedom. For the Lagrangian formulation to apply to systems with non-holonomic constraints we must work with more than \( N \) generalized coordinates and reduce the number of degrees of freedom down to \( N \) by imposing constraints.

We will explain the method of imposing non-holonomic constraints using the example of the rolling wheel introduced in lecture 10. This system has four degrees of freedom when we ignore the rolling-without-slipping constraint. We can describe the motion of this unconstrained wheel with four generalized coordinates: the position \((x, y)\) of the wheel’s center (in the plane where it moves), the angle \(\theta\) between the wheel’s axis and the \(x\)-axis, and the angle \(\phi\) the wheel has rotated about its axis.

Before we address the main problem of constraints, let’s construct the Lagrangian for the unconstrained (slipping) wheel. The kinetic energy has translational and rotational contributions:

\[
T_{\text{trans}} = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2) \quad T_{\text{rot}} = \frac{1}{2} (I \ddot{\theta}^2 + I_3 \ddot{\phi}^2).
\]

(16.1)

Because wheels are (rotationally) symmetric tops, we use the standard notation for the two principal moments of inertia. We will add a potential energy by tilting the \((x, y)\) plane by angle \(\alpha\) relative to the horizontal, where the \(y\)-axis corresponds to uphill:

\[
V = Mg \sin \alpha y.
\]

(16.2)

The unconstrained wheel therefore has Lagrangian

\[
L = T_{\text{trans}} + T_{\text{rot}} - V,
\]

(16.3)

and its action

\[
S[x, y, \theta, \phi] = \int_{t_1}^{t_2} L \, dt
\]

(16.4)

is a functional of four independent functions.

The rolling-without-slipping constraints are constraints on the velocities of the generalized coordinates (lecture 10):

\[
\dot{x} = r \dot{\phi} \sin \theta \quad \dot{\phi} = \frac{\dot{y}}{r \cos \theta}.
\]

(16.5)

(16.6)
We will interpret these as constraints on the set of possible variations that can be applied to the functions \(x(t), y(t), \theta(t)\) and \(\phi(t)\). There are no constraints among the values of these functions at any one particular time, say \(t = t_0\). However, consider the values at a short time later, say \(t' = t_0 + \Delta t\). We can write

\[
\begin{align*}
    x(t') &= x(t_0) + \delta x(t') \\
    y(t') &= y(t_0) + \delta y(t') \\
    \theta(t') &= \theta(t_0) + \delta \theta(t') \\
    \phi(t') &= \phi(t_0) + \delta \phi(t')
\end{align*}
\]

(16.7)

and ask what are the constraints on the changes to the generalized coordinates. Expressing the time derivatives in (16.5) as finite differences over time \(\Delta t\), we obtain the following two equations:

\[
\begin{align*}
    \delta x(t') - r \sin \theta(t') \delta \phi(t') &= 0 \\
    \delta y(t') + r \cos \theta(t') \delta \phi(t') &= 0.
\end{align*}
\]

(16.11, 16.12)

We should think of these as two linear constraints on the set of allowed variations at time \(t = t'\). The four variations at time \(t = t'\) in (16.7) are thus reduced to two independent variations. This has the effect of reducing four degrees of freedom to two.

To learn how to handle constraints of the type (16.11) in Hamilton’s variational principle, we consider an analogous problem for few variables. Suppose we have a function \(f(x, y)\) and wish to find “extreme” points \((x^*, y^*)\) of \(f\) in a restricted sense. Rather than insist that the change in \(f(x + \delta x, y + \delta y)\) vanishes to first order in \(\delta x\) and \(\delta y\) for arbitrary \(\delta x\) and \(\delta y\), we only require that this is true when these “variations” satisfy a linear constraint of the form

\[v_x(x, y) \delta x + v_y(x, y) \delta y = 0.\]

(16.13)

Here \(v = (v_x, v_y)\) is a vector field in the \((x, y)\) plane of our choosing. By exactly the same reasoning we used in lecture 13 to find extrema subject to constraint functions, we see that a necessary condition on the point \((x^*, y^*)\) is that the gradient of \(f\) at this point is parallel to \(v\), or

\[(\nabla f + \lambda v)|_{(x^*, y^*)} = 0\]

(16.14)

for some \(\lambda\). This reverts to the case of constraint functions when the vector field \(v\) can be expressed as the gradient of a scalar function — a constraint function \(g(x, y)\). But as you know, there are vector fields for which this is not possible. The difference between holonomic and non-holonomic constraints is therefore the same as the difference between vector fields that can and cannot be expressed as gradients of functions.

When there are multiple vector field constraints of the form (16.13), then just as in the case of multiple constraint functions (lecture 13) we look for extrema defined by

\[(\nabla f + \lambda_1 v_1 + \lambda_2 v_2 + \cdots)|_{(x^*, y^*)} = 0\]

(16.15)

The constraints on variations expressed by (16.11) correspond to \(2 \times \infty\) many vector fields — there being infinitely many instances of time \(t'\). We will focus on the second set of these vector fields, those involving \(y\). The vector field \(v(t)\) for the constraint at time \(t\) has only two non-zero components:

\[
\begin{align*}
    v(t)_{y(t)} &= 1 \\
    v(t)_{\phi(t)} &= r \cos \theta(t).
\end{align*}
\]

(16.16, 16.17)

We are now ready to solve the non-holonomic wheel problem. Using the rolling-without-slipping constraint (16.5), the translational kinetic energy takes the simple form

\[T_{\text{trans}} = \frac{1}{2} Mr^2 \dot{\theta}^2.\]

(16.18)
The Lagrangian for the unconstrained wheel is therefore

\[ L = \frac{1}{2}(Mr^2 + I_3)\dot{\phi}^2 + \frac{1}{2}I\dot{\theta}^2 - Mg \sin \alpha y. \] (16.19)

Notice that this does not involve \( x(t) \), so the action for the unconstrained wheel is a functional taking three functions as its argument:

\[ S[y, \theta, \phi] = \int_{t_1}^{t_2} L \, dt. \] (16.20)

The analog of the gradient \( \nabla f \) (for few variables) are the variational derivatives

\[ \frac{\delta S}{\delta y(t)} = -Mg \sin \alpha \] (16.21)

\[ \frac{\delta S}{\delta \theta(t)} = -\frac{d}{dt}(I\dot{\theta}) \] (16.22)

\[ \frac{\delta S}{\delta \phi(t)} = -\frac{d}{dt}\left((Mr^2 + I_3)\dot{\phi}\right). \] (16.23)

These should be set to zero for the unconstrained wheel. Because of the constraints, we need to include the “sum” over constraint vector fields

\[ \int_{t_1}^{t_2} \lambda(t')v(t') \, dt' \] (16.24)
in Hamilton’s principle, where the non-zero components of \( v(t) \) are given in (16.16). For example, the \( y(t) \) “gradient” component (16.21) appears in the integral (16.24) for \( t' = t \), where it is multiplied by \( \lambda(t) \). The modification to Hamilton’s principle for this component is therefore

\[ 0 = \frac{\delta S}{\delta y(t)} + \lambda(t) \cdot 1 = -Mg \sin \alpha + \lambda. \] (16.25)

Similarly, for the other components we obtain

\[ 0 = \frac{\delta S}{\delta \theta(t)} + \lambda(t) \cdot 0 = -I\ddot{\theta} \] (16.26)

\[ 0 = \frac{\delta S}{\delta \phi(t)} + \lambda(t) r \cos \theta(t) = -(Mr^2 + I_3)\ddot{\phi} + \lambda r \cos \theta. \] (16.27)

These three differential equations are easily solved. From (16.25) we find that the Lagrange multiplier function \( \lambda(t) \) is constant in time:

\[ \lambda = Mg \sin \alpha. \] (16.28)

Equation (16.26) implies that \( \theta(t) \) is a linear function:

\[ \theta(t) = \omega t + \theta_0. \] (16.29)

Finally, substituting this result and the Lagrange multiplier into (16.27), we obtain

\[ (Mr^2 + I_3)\ddot{\phi} = Mgr \sin \alpha \cos (\omega t + \theta_0). \] (16.30)

This is the same equation you obtained in the homework assignment by Newtonian methods.