Physics 3318: Analytical Mechanics	**Spring 2017**

Lecture 16: March 8

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16.1 Constrained variations in mechanics (continued)

16.1.1 Non-holonomic constraints: the rolling wheel

Recall that the Lagrangian formulation of mechanics is incomplete unless we can adapt the method to handle systems with non-holonomic constraints. In such systems it is impossible to specify the positions of all the particles with N generalized coordinates, where N is the number of degrees of freedom. For the Lagrangian formulation to apply to systems with non-holonomic constraints we must work with more than N generalized coordinates and reduce the number of degrees of freedom down to N by imposing constraints.

We will explain the method of imposing non-holonomic constraints using the example of the rolling wheel introduced in lecture 10. This system has four degrees of freedom when we ignore the rolling-without-slipping constraint. We can describe the motion of this unconstrained wheel with four generalized coordinates: the position (x, y) of the wheel's center (in the plane where it moves), the angle θ between the wheel's axis and the x-axis, and the angle ϕ the wheel has rotated about its axis.

Before we address the main problem of constraints, let's construct the Lagrangian for the unconstrained (slipping) wheel. The kinetic energy has translational and rotational contributions:

$$T_{\text{trans}} = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) \qquad T_{\text{rot}} = \frac{1}{2}(I\dot{\theta}^2 + I_3\dot{\phi}^2). \tag{16.1}$$

Because wheels are (rotationally) symmetric tops, we use the standard notation for the two principal moments of inertia. We will add a potential energy by tilting the (x, y) plane by angle α relative to the horizontal, where uphill corresponds to increasing y:

$$V = (Mg\sin\alpha)y. \tag{16.2}$$

The unconstrained wheel therefore has Lagrangian

$$\mathcal{L} = T_{\text{trans}} + T_{\text{rot}} - V, \tag{16.3}$$

and its action

$$S[x, y, \theta, \phi] = \int_{t_1}^{t_2} \mathcal{L}\, dt \tag{16.4}$$

is a functional of four independent functions.

The rolling-without-slipping constraints are constraints on the velocities of the generalized coordinates (lecture 10):

$$\begin{aligned} \dot{x} &= r\dot{\phi}\sin\theta \\ \dot{y} &= -r\dot{\phi}\cos\theta. \end{aligned} \tag{16.5}$$

We will interpret these as constraints on the set of possible variations that can be applied to the functions $x(t)$, $y(t)$, $\theta(t)$ and $\phi(t)$. There are no constraints among the values of these functions at any one particular time t. However, consider the values at a short time later, say $t' = t + \Delta t$. We can write

$$\begin{aligned} x(t') &= x(t) + \delta x(t) \\ y(t') &= y(t) + \delta y(t) \\ \theta(t') &= \theta(t) + \delta\theta(t) \\ \phi(t') &= \phi(t) + \delta\phi(t) \end{aligned} \tag{16.6}$$

and ask what are the constraints on the *changes* to the generalized coordinates. Expressing the time derivatives in (16.5) as finite differences over time Δt, we obtain the following two equations:

$$\begin{aligned} \delta x(t) - r\sin\theta(t)\,\delta\phi(t) &= 0 \\ \delta y(t) + r\cos\theta(t)\,\delta\phi(t) &= 0. \end{aligned} \tag{16.7}$$

We should think of these as two linear constraints on the set of allowed variations at time t. The four variations in (16.6) are thus reduced to two independent variations. This has the effect of reducing four degrees of freedom to two.

To learn how to handle constraints of the type (16.7) in Hamilton's variational principle, we consider an analogous problem for few variables. Suppose we have a function $f(x, y)$ and wish to find "extreme" points $(x^\star, y^\star)$ of f in a restricted sense. Rather than insist that the change in $f(x^\star + \delta x, y^\star + \delta y)$ vanishes to first order in δx and δy for arbitrary δx and δy, we only require that this is true when these "variations" satisfy a linear constraint of the form

$$v_x(x, y)\,\delta x + v_y(x, y)\,\delta y = 0. \tag{16.8}$$

Here $\mathbf{v} = (v_x, v_y)$ is a vector field in the (x, y) plane of our choosing. By exactly the same reasoning we used in lecture 13 to find extrema subject to constraint functions, we see that a necessary condition on the point $(x^\star, y^\star)$ is that the gradient of f at this point is parallel to $\mathbf{v}$, or

$$(\nabla f + \lambda\mathbf{v})|_{(x^\star, y^\star)} = 0 \tag{16.9}$$

for some λ. This reverts to the case of constraint functions when the vector field $\mathbf{v}$ can be expressed as the gradient of a scalar function — a constraint function $g(x, y)$. But as you know, there are vector fields for which this is not possible. The difference between holonomic and non-holonomic constraints is therefore the same as the difference between vector fields that can and cannot be expressed as gradients of functions.

When there are multiple vector field constraints of the form (16.8), then just as in the case of multiple constraint functions (lecture 13) we look for extrema defined by

$$(\nabla f + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots)|_{(x^\star, y^\star)} = 0. \tag{16.10}$$

Returning to the mechanics of the rolling wheel, we should start by taking stock of the *numbers* of things that enter into the Lagrange multiplier mathematics, now that we have the complication of variables and constraints at infinitely many times. First, the **number of dimensions** of the space we are working in is $4 \times \infty$, which you can think of as the functions $x(t)$, $y(t)$, $\theta(t)$ and $\phi(t)$ at infinitely many instants of time. The number of vector/gradient components in the counterpart to (16.10) is therefore $4 \times \infty$. Second, the

number of vector fields in the counterpart to (16.10) is $2 \times \infty$, one for each of the two constraints (16.7) at each instant of time. From the constraint equations (16.7) we see that the same time t applies to all of the variables in the constraint. We will therefore use the same symbol t to label variables and the constraint in which they reside. The names of the Lagrange multipliers for the constraints (16.7) will be $\lambda_1(t)$ and $\lambda_2(t)$.

Let's start with the $x(t)$-component of the vector expression (16.10). Remembering that the unconstrained action S corresponds to the function f being extremized, the gradient of f corresponds to the variational derivative

$$\frac{\delta S}{\delta x(t)} = -\frac{d}{dt}(M\dot{x}) = -M\ddot{x}. \tag{16.11}$$

From (16.7) we can read off the $x(t)$-components of the vector fields as the coefficients of $\delta x(t)$. There is just one term in the vector field sum of (16.10), because only the first constraint equation (and one time) involves $\delta x(t)$. The $x(t)$-component part of (16.10) is therefore the simple equation

$$-M\ddot{x} + \lambda_1(t) \cdot 1 = 0. \tag{16.12}$$

The other vector components are computed similarly:

$$-M\ddot{y} - Mg\sin\alpha + \lambda_2(t) = 0 \tag{16.13}$$

$$-I\ddot{\theta} = 0 \tag{16.14}$$

$$-I_3\ddot{\phi} - \lambda_1(t)r\sin\theta + \lambda_2(t)r\cos\theta = 0. \tag{16.15}$$

Exercise: Be sure to check these equations!

We have reduced the rolling wheel to the set of four equations above. Of course in solving these we may still use the original rolling-without-slipping equations (16.5). By taking the time derivative of the first and using (16.12), we obtain the following equation for one of the Lagrange multipliers:

$$\lambda_1(t) = M\ddot{x} \tag{16.16}$$

$$= Mr\left(\ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta\right). \tag{16.17}$$

Similarly, solving (16.13) for λ_2 and substituting $\ddot{y}$ from the time derivative of (16.5) we obtain

$$\lambda_2(t) = M\ddot{y} + Mg\sin\alpha \tag{16.18}$$

$$= Mr\left(-\ddot{\phi}\cos\theta + \dot{\phi}\dot{\theta}\sin\theta\right) + Mg\sin\alpha. \tag{16.19}$$

Substituting these two Lagrange multipliers into (16.15) we obtain (after some cancellation)

$$(I_3 + Mr^2)\ddot{\phi} = Mgr\sin\alpha\cos\theta. \tag{16.20}$$

Substituting the general solution of (16.14),

$$\theta(t) = \omega t + \theta_0, \tag{16.21}$$

into (16.20) we arrive at

$$(I_3 + Mr^2)\ddot{\phi} = Mgr\sin\alpha\cos(\omega t + \theta_0). \tag{16.22}$$

This is the same equation you will derive in the homework by Newtonian methods. Our solution of the rolling wheel problem is now complete. Two time integrals applied to (16.22) yields $\phi(t)$, and substituting it and our solution for $\theta(t)$ into (16.5) gives us $x(t)$ and $y(t)$ by a pair of integrals.