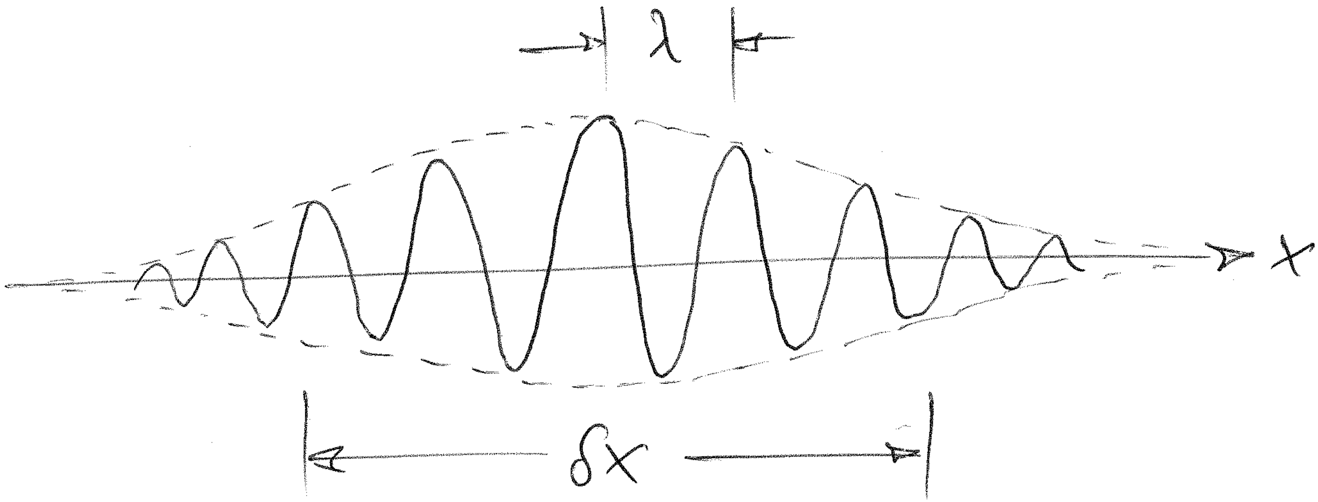
Lecture 15

wavepackets, in other contexts



both "wiggles" and "envelope" (dashed curve) move to the right (at velocities we will calculate at end of lecture)

in the case of a sound wave, passage of a wavepacket represents a musical tone, or "note"

- wavepacket size δx corresponds to <u>duration</u> of note

- wavepacket wavelength λ corresponds to <u>frequency</u> of note

turning δx and λ into time scales involves velocites, in fact two distinct velocites, as it turns out.

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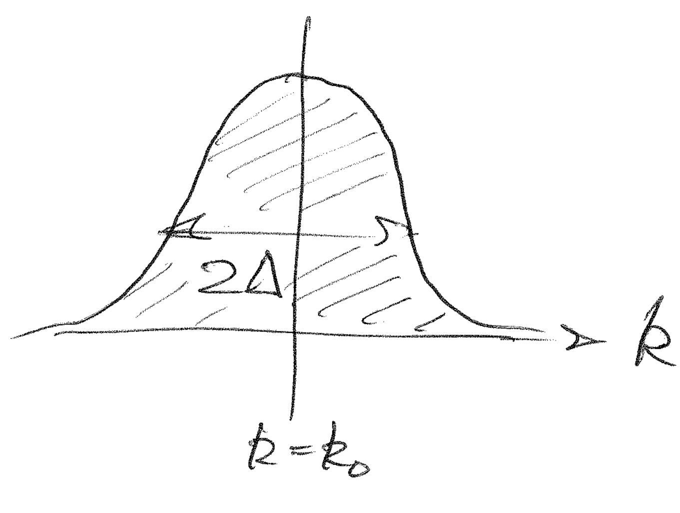
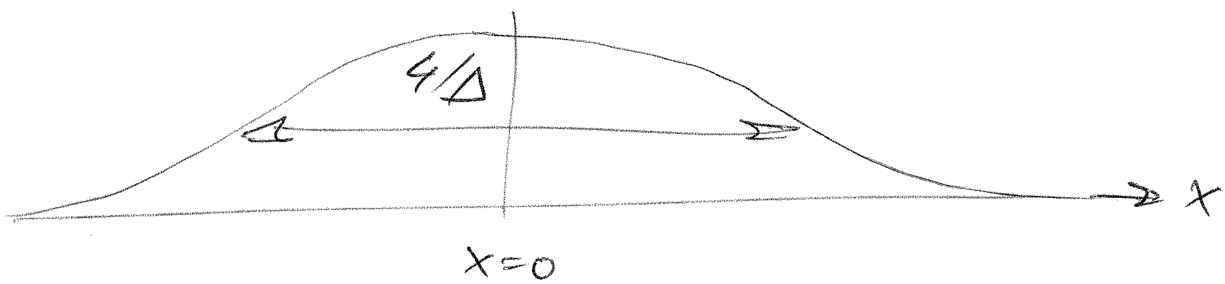
we formed the wavepacket by superposing cosine waves with this k-distribution:



enevelope of wavepacket :

$$H\, e^{-\frac{x^2}{(2/\Delta)^2}}$$



also a Gaussian.

in quantum mechanics particles are described by "probability amplitude waves", wave packets are a bridge to classical physics, in the following sense :

"uncertainty" of particle position

$$\parallel$$

size of wavepacket

$$\delta x \simeq 2/\Delta$$

particle momentum (De Broglie relationship):

$$p = \frac{h}{\lambda} = \frac{2\pi}{\lambda}\, h/2\pi = k\hbar$$

$\left(\begin{array}{c} \hbar = h/2\pi \\ \text{Planck's const.} \\ \text{over } 2\pi \end{array} \right)$

momentum uncertainty:

$$\delta p = \underbrace{\delta k}_{\substack{k\text{-distribution} \\ \text{to construct wavepacket}}} \cdot \hbar = \Delta \cdot \hbar$$

Heisenberg observed,

$$\delta x \cdot \delta p = 2\hbar,$$

so uncertainties cannot <u>both</u> be small — unlike classical physics where $\delta x = 0$ and $\delta p = 0$.

Actual "Heisenberg uncertainty relation" is an inequality, and wavepackets represent the "best one can do" (inequality $\rightarrow$ equality).

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superpose running waves at arbitrary times:

$$S(x,t) = \int_{-\infty}^{\infty} h(k) \cos(kx - \omega(k)t)\, dk$$

as before, $h(k) = \frac{H}{\sqrt{\pi}} \frac{e^{-(k-k_0)^2/\Delta^2}}{\Delta}$

consider limit $\Delta \rightarrow 0$, so only very small range of k's contribute.

(15.6)

approximate $\omega(k)$ for

k's in small range about $k = k_0$:

$$\omega(k) = \underbrace{\omega(k_0)}_{\omega_0} + \underbrace{\omega'(k_0)}_{\omega_0'}(k-k_0) + \cdots$$

ignore quadratic & higher terms:

$$\omega(k) \cong \omega_0 + \omega_0'(k-k_0)$$

Repeat previous calculation ($t=0$):

$$S(x,t) = \frac{H}{\sqrt{\pi}} \, \mathrm{Re}\left[\int_{-\infty}^{\infty} e^{-\frac{(k-k_0)^2}{\Delta^2} + i\left(kx - \omega_0 t - \omega_0'(k-k_0)t\right)} \frac{dk}{\Delta} \right]$$

exponent:

$$-\frac{(k-k_0)^2}{\Delta^2} + i\Big(\underbrace{k_0 x - \omega_0 t}_{\text{in } t=0 \text{ calc. this was } k_0 x} + (k-k_0)\underbrace{(x-\omega_0' t)}_{\text{in } t=0 \text{ calc. this was } x}\Big)$$

previous calculation with these replacements:

$$S(x,t) = H \underbrace{e^{-\frac{\Delta^2}{4}(x-\omega_0' t)^2}}_{\text{Gaussian envelope with center at } \omega_0' t} \underbrace{\cos(k_0 x - \omega_0 t)}_{\text{"wiggles"}}$$

<u>envelope</u> (Gauss center) moves with "group velocity":

$$V_g = \omega_0' = \left.\frac{d\omega}{dk}\right|_{k=k_0}$$

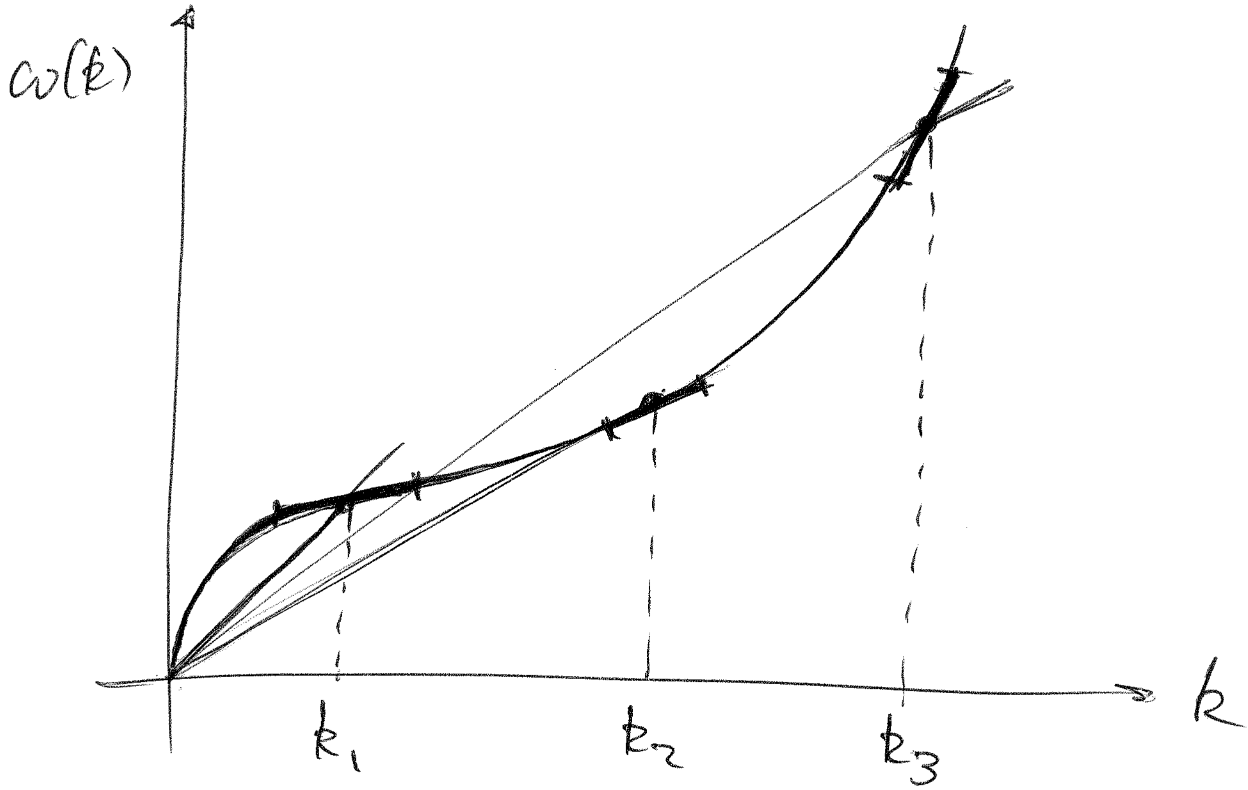
<u>wiggles</u> move with "phase velocity":

$$V_\phi = \frac{\omega_0}{k_0} = \frac{\omega(k_0)}{k_0}$$

these are, respectively, <u>tangent</u> and <u>chord</u> slopes of dispersion curve $\omega(k)$.

surface-wave dispersion

(15.4)



k_1 : $\frac{d\omega}{dk} < \frac{\omega}{k}$ $\quad V_g < V_\phi$

k_2 : $\frac{d\omega}{dk} = \frac{\omega}{k}$ $\quad V_g = V_\phi$

k_3 : $\frac{d\omega}{dk} > \frac{\omega}{k}$ $\quad V_g > V_\phi$