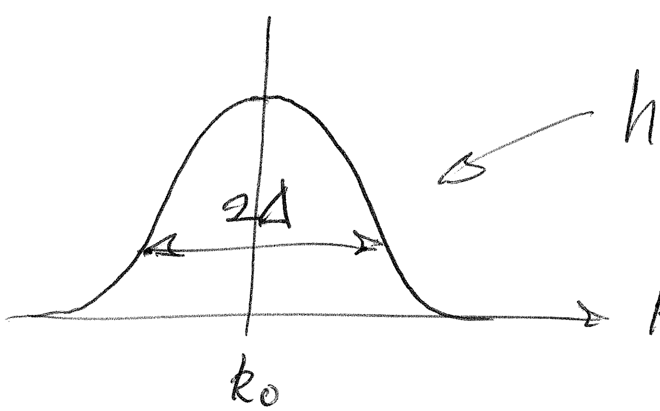
Lecture 14

The Gaussian wavepacket, I

set $t=0$ for now; will return to $t \neq 0$ later.

$$S(x,0) = \int_{-\infty}^{\infty} h(k) \cos kx \, dk$$

$h(k)$ = wave-vector distribution in our superposition



$$h(k) = \frac{1}{\sqrt{\pi}} \frac{H}{\Delta} e^{-(k-k_0)^2/\Delta^2}$$

k_0 = mean ("central") wave-vector

Δ = width of distribution

(we will generally be interested in $\Delta \to 0$ limit)

"Normalization" of distribution:

$$\int_{-\infty}^{\infty} h(k)\,dk = \frac{H}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(k-k_0)^2/\Delta^2} \frac{dk}{\Delta}$$

change of integration variable:

$$\frac{k-k_0}{\Delta} = q \qquad \frac{dk}{\Delta} = dq$$

$$\int_{-\infty}^{\infty} h(k)\,dk = \frac{H}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2}\,dq = H$$

(normalization is independent of width Δ)

$$S(x,0) = \int_{-\infty}^{\infty} h(k) \cos kx \; dk$$

try to write entirely in terms of exponential:

$$\cos kx = \mathrm{Re}\left[e^{ikx} \right]$$

$$h(k) \cos kx = \mathrm{Re}\left[h(k)\, e^{ikx} \right]$$

$$(\text{since } h(k) = \text{real})$$

can take real part after doing integral:

$$S(x,0) = \mathrm{Re}\left[\int_{-\infty}^{\infty} h(k)\, e^{ikx} \, dk \right]$$

$$= \frac{H}{\sqrt{\pi}} \mathrm{Re}\left[\int_{-\infty}^{\infty} e^{-\frac{(k-k_0)^2}{\Delta^2} + ikx} \frac{dk}{\Delta} \right]$$

rewrite stuff in exponent:

$$-\frac{(k-k_0)^2}{\Delta^2}+ikx = -\frac{(k-k_0)^2}{\Delta^2}+i(k-k_0)x + ik_0x$$

$$\left(\frac{k-k_0}{\Delta}=q\right) = -q^2+ix\Delta q+ik_0x$$

complete-the-square:

$$-q^2+ix\Delta q = -\left(q-\frac{i}{2}x\Delta\right)^2-\frac{x^2\Delta^2}{4}$$

$$\text{exponent} = -\left(q-\tfrac{i}{2}x\Delta\right)^2 \underbrace{-\frac{x^2\Delta^2}{4}+ik_0x}_{\text{independent of } q \text{ (integration)}}$$

back to integral:

$$S(x,0)=\frac{H}{\sqrt{\pi}}\,\mathrm{Re}\left[e^{-\frac{x^2\Delta^2}{4}+ik_0x}\int_{-\infty}^{\infty}e^{-\left(q-\frac{i}{2}x\Delta\right)^2}dq\right]$$

$$\int_{-\infty}^{\infty} e^{-(q-\frac{i}{2}x\Delta)^2}\, dq = \sqrt{\pi}$$

(shift in integration variable, $q' = q - \frac{i}{2}x\Delta$, can be <u>complex</u>; wait for your course in complex analysis to get the proper justification)

$$S(x,0) = H \, \mathrm{Re}\left[e^{-\frac{1}{4}x^2\Delta^2 + ik_0x} \right]$$

$$= H e^{-\frac{1}{4}x^2\Delta^2} \cos(k_0 x)$$