Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 14: March 3

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14.1 Calculus of variations with constraints

14.1.1 One constraint: the hanging chain

A good strategy for understanding how to implement constraints with Lagrange multipliers in problems with infinitely many variables — and possibly infinitely many constraints — is to generalize from the few variable/constraint examples we worked out in lecture 13. By replacing functions with functionals, partial derivatives (gradients) with variational derivatives, sums with integrals, etc., the method is no different from the few-variable case. If this bothers you, then consider approximating your functional by a sum of finitely many terms (*i.e.* sampling the function at evenly spaced positions in space).

Suppose we attach the ends of a fine chain of length l_1 at two points with horizontal separation l_2 and heights $y(0) = y_1$, $y(l_2) = y_2$. What shape $y(x)$ gives the chain the lowest possible gravitational energy?

This minimization problem involves two functionals:

$$f[y] = \text{potential energy of curve } y(x) \tag{14.1}$$
$$g[y] = \text{length of curve } y(x). \tag{14.2}$$

Our names for these functionals are the same names we used in the finitely-many variable examples of lecture 13: f is the function/functional being minimized and g the function/functional whose value is constrained. Both of these functionals are integrals:

$$f[y] = \int dm\, gy = \int_0^{l_2} \mu g y \sqrt{1 + y'^2}\, dx \tag{14.3}$$
$$g[y] = \int_0^{l_2} \sqrt{1 + y'^2}\, dx. \tag{14.4}$$

Here μ is the (uniform) mass density of the chain and $y' = dy/dx$. The counterpart of the condition

$$0 = (\nabla f + \lambda \nabla g)\big|_{(x^\star, y^\star)} \tag{14.5}$$

for finitely-many variables is the condition

$$0 = \left(\frac{\delta f}{\delta y(x)} + \lambda \frac{\delta g}{\delta y(x)} \right)\Bigg|_{y^\star}, \tag{14.6}$$

where the variational derivatives are evaluated at the minimum-energy/given-length curve $y^\star(x)$. In addition to having to solve this infinite system of equations for the infinitely many variables $y(x)$ (*i.e.* solving a differential equation), we also have to satisfy the single equation

$$g[y] = l_1. \tag{14.7}$$

As in the finitely-many variable case we have the right number of equations to determine a unique $y^\star(x)$. We begin by solving the differential equation (14.6) and find curves $y(x)$ that depend on the value of the Lagrange multiplier λ. Substituting these into (14.7) will then determine the value of λ that gives the correct chain length.

We will solve (14.6) by recalling something we learned in connection with the Hamiltonian (lecture 9). Let's define the functional

$$S[y] = f[y] + \lambda g[y] = \int_0^{l_2} \mathcal{L}(y, y', x)dx, \tag{14.8}$$

where our symbols S and $\mathcal{L}$ were chosen to remind us of the corresponding functionals/functions in mechanics (action and Lagrangian). The function that corresponds to the Lagrangian is

$$\mathcal{L}(y, y', x) = (\mu g y + \lambda)\sqrt{1 + y'^2}. \tag{14.9}$$

Note that this "Lagrangian" has no explicit x and should lead to the same properties we have when the Lagrangian of mechanics has no explicit t. Recall that when t is absent from the Lagrangian the Hamiltonian is constant. The same should be true for the "Hamiltonian" constructed from (14.9), with x derivatives replacing t derivatives:

$$\begin{aligned} \mathcal{H} &= y'\frac{\partial \mathcal{L}}{\partial y'} - \mathcal{L} && (14.10) \\ &= y'(\mu g y + \lambda)\frac{y'}{\sqrt{1+y'^2}} - (\mu g y + \lambda)\sqrt{1+y'^2}. && (14.11) \end{aligned}$$

Since S has the units of f or energy, $\mathcal{L}$ and therefore also $\mathcal{H}$ has units of energy per unit length. Let's let ϵ be the constant value of $\mathcal{H}$, and simplify the equation $\mathcal{H} = \epsilon$:

$$\left(\frac{\mu g y + \lambda}{\epsilon}\right)^2 = 1 + y'^2. \tag{14.12}$$

This is a first order differential equation, significantly easier to solve than the second order differential equation (Euler-Lagrange) we get from $\delta S/\delta y(x) = 0$ (14.6).

An efficient way to solve (14.12) is to start by rescaling and shifting variables. Since $\epsilon/(\mu g)$ has units of length, we can define a rescaled and dimensionless position by

$$\tilde{x} = \left(\frac{\mu g}{\epsilon}\right) x. \tag{14.13}$$

Now use the same length scale to define a dimensionless height variable:

$$\tilde{y} = \frac{\mu g\, y + \lambda}{\epsilon}. \tag{14.14}$$

Since $d\tilde{y}/d\tilde{x} = dy/dx$, equation (14.12) takes the following very simple form,

$$\tilde{y}^2 = 1 + \tilde{y}'^2, \tag{14.15}$$

with general solution

$$\tilde{y}(\tilde{x}) = \pm \cosh{(\tilde{x} - \tilde{x}_0)}. \tag{14.16}$$

Using (14.14) to express $y(x)$ in terms of $\tilde{y}(\tilde{x})$, we obtain

$$y(x) = -\frac{\lambda}{\mu g} \pm \frac{\epsilon}{\mu g} \cosh\left(\frac{\mu g}{\epsilon} x - \tilde{x}_0\right). \tag{14.17}$$

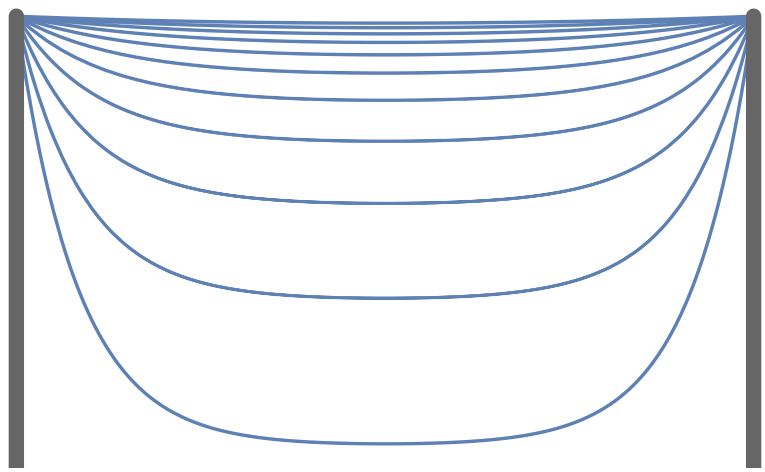
The $\pm$ sign in the second term can be absorbed by a change of sign in the arbitrary parameters ϵ and $\tilde{x}_0$. We now have the right number of constants (λ, ϵ, $\tilde{x}_0$) for satisfying the three equations

$$y(0) = y_1 \tag{14.18}$$

$$y(l_2) = y_2 \tag{14.19}$$

$$\int_0^{l_2} \sqrt{1 + y'^2} dx = l_1. \tag{14.20}$$

The hyperbolic cosine function is sometimes referred to as the "catenary curve" because it describes the shape of a hanging chain.



Equilibrium (lowest energy) shapes of a chain hung between equal-height supports for various lengths of chain.

Question: Describe qualitatively the shape of the hanging chain as the chain length becomes very much greater than the distance between the supports (and the lowest point still clears the ground).