Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 13: February 27

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13.1 Extensions of the variational calculus

13.1.1 Multiple degrees of freedom

When our system has only holonomic constraints and N degrees of freedom, we can construct a Lagrangian $\mathcal{L} = T - V$ of the form

$$\mathcal{L} = \mathcal{L}(q_1, \ldots, q_N\,;\, \dot{q}_1, \ldots, \dot{q}_N; t). \tag{13.1}$$

As in systems with one degree of freedom (lecture 11) we define the action as the time integral of $\mathcal{L}$:

$$S[q_1, \ldots, q_N] = \int_{t_1}^{t_2} \mathcal{L}\, dt. \tag{13.2}$$

The action is now a functional of N functions. When calculating the variational derivative with respect to one of the q's we can treat the other functions as fixed (in complete analogy with the definition of partial derivatives). The calculation of

$$\frac{\delta}{\delta q_k(t)}\, S = \frac{\partial \mathcal{L}}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right), \tag{13.3}$$

is really no different from the calculation in lecture 11 for a single degree of freedom. We see that the condition for S to be extremal with respect to variations of any of the generalized coordinates q_k, and at any time t, is just the full set of Euler-Lagrange equations for the system.

Continuum systems are systems whose degrees of freedom are huge in number and naturally described by a continuous parameter. Consider an elastic string that we describe with a displacement $y(x, t)$. We can think of the position x as a parameter or label for the degrees of freedom, much like the label $k = 1, \ldots, N$ we have been using up to now for systems with few degrees of freedom. All of what we have discussed about systems with few degrees of freedom continue to hold for continuum systems when we treat the position of the degree of freedom as a label.

Because the degree-of-freedom labels in continuum systems correspond to position, the structures of their Lagrangians usually have the form of integrals over position and sometimes involve derivatives with respect to position. For example, here are the kinetic and potential energies of the elastic string (μ = mass density, τ = tension, $y' = dy/dx$):

$$T = \int \frac{1}{2}\mu\, \dot{y}^2\, dx, \qquad V = \int \frac{1}{2}\tau\, y'^2\, dx. \tag{13.4}$$

Breaking the string up into segments of length dx, we recognize the expression for T as the sum of kinetic energies of particles of mass $\mu\, dx$. Similarly, V is the elastic energy when the same set of particles is joined by elastic springs.

In view of the fact that both T and V are integrals, we define the *Lagrangian density*

$$\rho(y, \dot{y}, y') = \frac{1}{2}\mu\,\dot{y}^2 - \frac{1}{2}\tau\,y'^2, \tag{13.5}$$

whose integral $\int \rho\,dx$ is the Lagrangian $\mathcal{L} = T - V$. The action is therefore a two-dimensional integral:

$$S[y] = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx\,\rho(y, \dot{y}, y'). \tag{13.6}$$

Hamilton's principle holds for this continuum system as it does for any other — the large number of degrees of freedom changes nothing. The action is extremal when its variational derivative vanishes at all t and for all x. When calculating the variational derivative we must perform an integration by parts for the x integral (in addition to the t integral) in order to end up with

$$\delta S = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx\,(\cdots)\,\delta y(x,t). \tag{13.7}$$

We also need to restrict to variations $\delta y(x,t)$ that vanish on a boundary in the (x,t) plane to avoid endpoint terms in the integration by parts. With these conditions in place, the expression $(\cdots)$ above defines the variational derivative of the action and its form should by now be familiar:

$$(\cdots) = \frac{\delta S}{\delta y(x,t)} = \frac{\partial \rho}{\partial y} - \frac{d}{dt}\left(\frac{\partial \rho}{\partial \dot{y}}\right) - \frac{d}{dx}\left(\frac{\partial \rho}{\partial y'}\right). \tag{13.8}$$

From the Lagrangian density of the string (13.5), we obtain

$$\frac{\partial \rho}{\partial y} = 0, \qquad \frac{\partial \rho}{\partial \dot{y}} = \mu \dot{y}, \qquad \frac{\partial \rho}{\partial y'} = -\tau y', \tag{13.9}$$

for the three partial derivatives in (13.8). Setting $\delta S/\delta y = 0$ (Hamilton's principle), we get the following equations of motion:

$$0 = -\mu \ddot{y} + \tau y''. \tag{13.10}$$

This is really a "continuum system" of equations of motion (one per position x), also known as the wave equation

$$\ddot{y} - v^2 y'' = 0, \tag{13.11}$$

with wave velocity $v = \sqrt{\tau/\mu}$.

Continuum systems are ubiquitous in Nature and their dynamics are neatly encoded in a single expression, their Lagrangian density. The Lagrangian density for electromagnetic waves is $E^2 - B^2$, where the reversal of the sign relative to the energy density, $E^2 + B^2$, is analogous to the transformation of $\mathcal{L} = T - V$ into $\mathcal{H} = T + V$. This is a density in three-dimensional space, so the action S for electromagnetic waves is an integral over four-dimensional space-time. Of course both $\mathbf{E}$ and $\mathbf{B}$ must be expressed in terms of the same set of generalized coordinates in order to be able to use the Lagrangian formalism. You might recall that this is accomplished by the vector potential:

$$\mathbf{E} = -\dot{\mathbf{A}}, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{13.12}$$

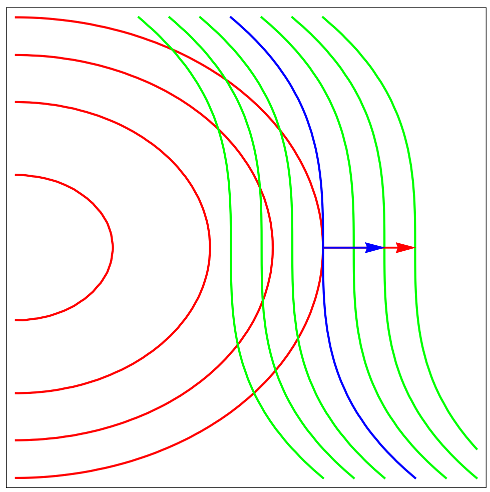
The Lagrangian picture of electromagnetic waves is therefore the dynamics of the vector potential field $\mathbf{A}$, where the electric energy density E^2 corresponds to its kinetic energy and the magnetic energy density B^2 its potential energy. There is a complication in applying the Lagrangian formalism to the dynamics of the vector potential because there are actually fewer degrees of freedom as a result of "gauge-fixing". This complication also arises in systems with finitely many degrees of freedom and brings us to our next topic.

13.1.2 Finding extrema subject to constraints

It may happen that our Lagrangian is much easier to express with more variables than the number of degrees of freedom, or that an expression having the correct number of variables is impossible because the constraints are non-holomorphic. The method of *Lagrange multipliers* is the general calculus procedure for dealing with this situation.

To see how Lagrange multipliers apply in mechanics we first review the method for functions of two and three variables. We then apply the method to the hanging chain problem, a problem with a single constraint and infinitely many variables (the shape of the curve formed by the chain).

Suppose we wish to minimize $f(x, y)$ subject to the constraint that $g(x, y) = c$. A condition on the minimum $(x^\star, y^\star)$ is made transparent in a plot where we superimpose the contours of the two functions f and g.



Contours of the function $f(x, y)$ to be minimized (red) and of the constraint function $g(x, y)$ (green and blue). The blue contour corresponds to $g = c$ and its point of tangency with one of the red contours, at $(x^\star, y^\star)$, is the minimum of f subject to the constraint $g = c$. The gradients of the two functions are parallel at the point of tangency.

If the g-contour through $(x^\star, y^\star)$ were not parallel to the f-contour, then the value of f could be further minimized by moving along the g-contour (the direction given by the sense of the change in f). A necessary condition on $(x^\star, y^\star)$ is therefore that the two equations (one for each vector component)

$$0 = (\nabla f + \lambda \nabla g)|_{(x^\star, y^\star)} \tag{13.13}$$

can be solved for some λ (the Lagrange multiplier). Since we also have the constraint equation

$$c = g(x^\star, y^\star), \tag{13.14}$$

we have altogether $2 + 1 = 3$ equations that will determine the three numbers $(x^\star, y^\star, \lambda)$.

Now suppose we wish to minimize $f(x, y, z)$ subject to the constraints $g_1(x, y, z) = c_1$ and $g_2(x, y, z) = c_2$. Just as in the previous problem, a condition on any minimum $(x^\star, y^\star, z^\star)$ involves the gradients

$$\mathbf{v}_1 = \nabla g_1(x^\star, y^\star, z^\star), \qquad \mathbf{v}_2 = \nabla g_2(x^\star, y^\star, z^\star). \tag{13.15}$$

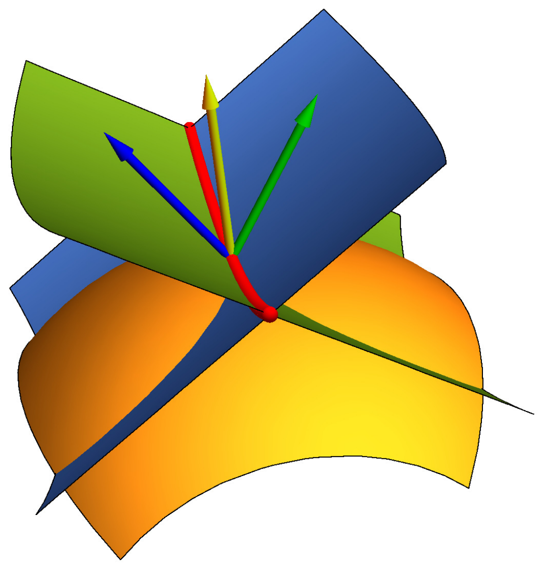
Both constraints, $g_1(x,y,z) = c_1$ and $g_2(x,y,z) = c_2$, are satisfied on a curve that passes through $(x^\star, y^\star, z^\star)$, and both vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, are perpendicular to this curve at the point $(x^\star, y^\star, z^\star)$. Now if f is a minimum on this curve at $(x^\star, y^\star, z^\star)$, its gradient must also be perpendicular to the curve. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ span the plane perpendicular to the curve at $(x^\star, y^\star, z^\star)$, it must be true that

$$0 = (\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2)\big|_{(x^\star, y^\star, z^\star)} \tag{13.16}$$

can be solved for suitable λ_1 and λ_2. This set of three equations (one for each component) together with the two constraint equations

$$c_1 = g_1(x^\star, y^\star, z^\star), \qquad c_2 = g_2(x^\star, y^\star, z^\star), \tag{13.17}$$

determine the five numbers $(x^\star, y^\star, z^\star, \lambda_1, \lambda_2)$.



Minimizing a function $f(x,y,z)$ subject to constraints $g_1(x,y,z) = c_1$ (green surface) and $g_2(x,y,z) = c_2$ (blue surface). Both constraints are satisfied on the red curve, where the two constraint surfaces intersect. The yellow surface is the level-set $f(x,y,z) = f_{\text{min}}$ that is tangent to the red curve. Because the gradients of f (yellow vector), g_1 (green vector), and g_2 (blue vector) are all perpendicular to the red curve, they must be co-planar and therefore satisfy an equation of the form (13.16).