Lecture 12: February 24

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12.1 Hamilton's principle

By a strange accident, the differential equation for the curve $y(x)$ in the brachistochrone problem has the same structure, when expressed in terms of the function $F(y, y', x)$, as the differential equation for a generalized coordinate $q(t)$ when expressed in terms of the Lagrangian $\mathcal{L}(q, \dot{q}, t)$:

$$0 = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \tag{12.1}$$

$$0 = \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right). \tag{12.2}$$

When this was noticed by Hamilton in the 19^{th} century he realized that an alternative to the Euler-Lagrange statement of the laws of mechanics (12.2) is the statement

$$0 = \frac{\delta}{\delta q(t)} S[q], \tag{12.3}$$

where $S[q]$ is the functional

$$S[q] = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t) dt. \tag{12.4}$$

This functional, called the *action*, is defined for all conceivable trajectories $q(t)$ that join an initial coordinate value $q(t_1)$ to a final coordinate value $q(t_2)$ — not just the actual trajectory.

To appreciate the strangeness of Hamilton's principle, consider a mass that is tossed upward from $z = 0$ at time $t = 0$, and caught at the same height after time T, so $z(0) = z(T) = 0$[1]. There are infinitely many trajectories that satisfy those end-point conditions, including very strange ones such as where the mass instantaneously jumps to some height $z = h$, hovers there for a time T, and then just as abruptly jumps back down to $z = 0$. Hamilton's principle rejects all those strange trajectories because they do not have property (12.3) — their action is not extremal. The extremal-action trajectory selected by Hamilton's principle is the actual trajectory, $z^\star(t) = g\, t(T - t)/2$.

Thinking back to the brachistochrone problem, our first reaction to Hamilton's principle is that Nature, for whatever reason, is trying to minimize action. However, already the case of the simple harmonic oscillator shows that this is not what Hamilton's principle is about. In a homework problem you will see that there are infinitely many modes of perturbation of the harmonic oscillator trajectory where the action is a local maximum, just as there are infinitely many modes where it is a local minimum. A better characterization of the extremal-action trajectory is therefore a "saddle".

[1]The height z of the mass is the generalized coordinate q in this example.

12.1.1 Classical mechanics as an approximation to quantum mechanics

The true laws of mechanics — at least as we understand them today — are expressed in the language of quantum mechanics. Classical mechanics emerges as an approximation of the equations of quantum mechanics. The significance of the Lagrangian formulation of classical mechanics — over the Newtonian formulation — is that the nature of this approximation is made very explicit.

Quantum mechanics can also be formulated in terms of the Lagrangian and action we define in classical mechanics. There is really only one difference: rather than singling out just the extremal-action trajectories, quantum mechanics uses in an essential way all conceivable trajectories connecting the endpoints of the motion. Interestingly, quantum mechanics does not assign any special importance (e.g. weighting factors) to the extremal trajectories. In fact, quantum mechanics assigns "amplitudes" to trajectories in the most democratic manner possible: a complex number of unit magnitude (phasor), $e^{i\phi}$. The phase of the amplitude is the following functional:

$$\phi[q] = S[q]/\hbar = \frac{1}{\hbar}\int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)dt. \tag{12.5}$$

The appearance of Planck's constant is both appropriate (in a quantum mechanical formula) and points out the fact that action has units of energy$\times$time (so that ϕ is properly dimensionless).

In quantum mechanics the net amplitude for the system to propagate between $q = q_1$ at $t = t_1$, and $q = q_2$ at $t = t_2$, is expressed as the single complex number given by the sum of phasors over all trajectories that join these endpoints:

$$\mathrm{Amp}(1 \to 2) = \sum_{q(t)} e^{i\phi[q]}. \tag{12.6}$$

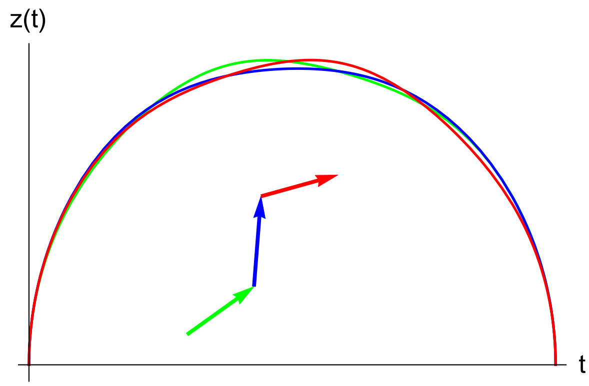
In the complex plane we represent phasors by vectors of unit magnitude. The quantum amplitude (12.6) is the sum of infinitely many such vectors. If the action $S[q]$ varies rapidly on the scale of $\hbar$ from one trajectory to a nearby trajectory, then our picture of the phasor sum is that of a random walk. This would lead us to expect a small net amplitude. But if we recall Hamilton's principle, we realize there must always be a subset of trajectories (still an infinite number) whose phasors are almost perfectly parallel and combine to give a large net amplitude.

Trajectories close to the action-extremizing "classical" trajectory $q^\star$ are special because the variation of $S[q]$ vanishes to first order in any perturbation $\delta q(t)$. The phase angles of the corresponding phasors will therefore be nearly equal. Perturbations of non-extremal trajectories are not so fortunate — the corresponding phasors will not be nearly as parallel.

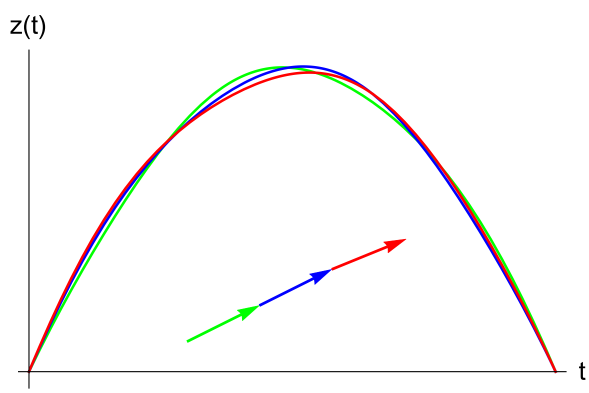
The near parallelism of phasors for trajectories near $q^\star$ is the same effect as constructive interference in optics. The contribution to the phasor sum from these trajectories will be very large and overwhelm the random-walk contribution from the other trajectories, those not near $q^\star$. A good approximation of the quantum mechanical sum (12.6) is therefore to limit the sum to only those trajectories near the action-extremizing trajectory $q^\star$. Classical mechanics is said to be the $\hbar \to 0$ limit of quantum mechanics because

$$\phi[q] = S[q]/\hbar \tag{12.7}$$

puts increasingly strong demands on the proximity of trajectories to $q^\star$, as $\hbar \to 0$, in order that the phasors remain parallel. But that is, in a sense, how we think about "classical mechanics": motion with a single, unique trajectory. A graphical comparison of the two kinds of trajectories (non-extremal vs. extremal) and their corresponding phasors is shown on the next page.



Three nearby trajectories that are far from the true trajectory of a mass thrown vertically in a gravitational field. Because the action S is not extremal, it varies linearly in perturbations of a trajectory. The angle of the corresponding phasor, $\phi = S/\hbar$, therefore also varies linearly and can be large even for small perturbations because $\hbar$ is so small.



Three trajectories near the classical trajectory (a parabola) where the action is extreme and has zero linear variation with perturbations. The corresponding phasor angles are therefore much less sensitive to perturbations, with the result that a very great number of them (in the sum over all trajectories) combine coherently as shown.