11.1 The calculus of variations

As physicists/mathematicians at the end of the 17th century began exploring the analytical powers made possible by calculus, a particular problem was posed by Johann Bernoulli that would provide the foundation a century later for a new principle of mechanics. We will repeat this historical line of development, starting with Bernoulli’s brachistochrone problem and the calculus of variations it inspired, then applying the result to mechanics where it is now known as Hamilton’s principle.

11.1.1 The brachistochrone problem

Bernoulli asked for the curve \( y(x) \) in a vertical plane along which a particle of mass \( M \), acted on only by gravity (no friction), is transported in the shortest time between endpoints \( (x = 0, y = 0) \) and \( (x = l, y = 0) \), starting/arriving with zero velocity. Using elementary mechanics we can derive a formula for the time of transit \( \Delta t \) for a general curve \( y(x) \) having the given endpoints.

By conservation of energy, the speed \( v(y) \) of the mass satisfies

\[
\frac{1}{2} M v^2 + Mgy = 0, \tag{11.1}
\]

with solution

\[
v(y) = \sqrt{-2gy}. \tag{11.2}
\]

The arc-length \( ds \) traversed by the mass along the curve during time \( dt \) is

\[
ds = v dt, \tag{11.3}
\]

and can be expressed as

\[
ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2} \, dx, \tag{11.4}
\]

where \( y' \) is our shorthand for the derivative \( dy/dx \). Solving for \( dt \) in (11.3) in terms of \( ds \) from (11.4) and \( v \) from (11.2), we obtain

\[
\Delta t = \int dt = \int_0^l \sqrt{\frac{1 + y'^2}{-2gy}} \, dx \tag{11.5}
\]

\[
= \int_0^l F(y, y', x) \, dx. \tag{11.6}
\]

The function \( F \) does not actually depend on \( x \), but we will include that in our general discussion since we later apply the principle to mechanics, where the Lagrangian \( L(q, \dot{q}, t) \) takes the place of \( F \).
11.1.2 The variational derivative

To solve the brachistochrone problem we want to minimize the elapsed time “functional”,

\[ \Delta t[y] = \int_{0}^{l} F(y, y', x) \, dx, \tag{11.7} \]

with respect to all possible smooth curves \( y(x) \) that join the given endpoints. We refer to \( \Delta t \) as a functional (rather than a function) to emphasize the fact that its value depends on a function (rather than a finite number of variables). In the following discussion we will not make use of the fact that

\[ F(y, y', x) = \sqrt{\frac{1 + y'^2}{-2gy}}, \tag{11.8} \]

other than the fact that \( F \) has the three arguments \( y, y', \) and \( x \).

Suppose that \( y^*(x) \) is the function that minimizes \( \Delta t \). If that is the case, then it should not be possible to slightly perturb \( y^* \) and thereby decrease \( \Delta t \). With that in mind, let’s calculate \( \Delta t \) when the argument of this functional is \( y^* + \delta y \), where \( \delta y(x) \) is an infinitesimally small function:

\[ \Delta t[y^* + \delta y] = \int_{0}^{l} \left( F[y^*] + \left. \frac{\partial F}{\partial y'} \right|_{y^*} \delta y + \left. \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right|_{y^*} \delta y + O(\delta y^2) \right) \, dx. \tag{11.9} \]

The notation \( \big|_{y^*} \) indicates that the preceding expression (\( F, \partial F/\partial y', \) etc.) is to be evaluated with the function \( y^* \). Integrating by parts the third term of (11.9),

\[ \int_{0}^{l} \left( \frac{\partial F}{\partial y'} \right|_{y^*} \delta y + \left. \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right|_{y^*} \delta y \bigg|_{x=0} = - \int_{0}^{l} \left( \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right|_{y^*} \delta y \bigg|_{x=0} \]. \tag{11.10} \]

The second term vanishes because the perturbation must vanish at the endpoints (\( \delta y(0) = \delta y(l) = 0 \)) in order that the curve still joins the original points of the brachistochrone problem. Subtracting \( \Delta t[y^*] \) from (11.9) and using the result of the partial integration (11.10), we obtain

\[ \Delta t[y^* + \delta y] - \Delta t[y^*] = \int_{0}^{l} \left( \left( \frac{\partial F}{\partial y'} \right|_{y^*} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right|_{y^*} \delta y + O(\delta y^2) \right) \, dx \]

\[ = \int_{0}^{l} (G(x)\delta y(x) + O(\delta y^2)) \, dx. \tag{11.11} \]

Now, if the function \( G(x) \) that multiplies the perturbation \( \delta y(x) \) is non-zero in some small interval, then by making \( \delta y(x) \) non-zero and of the correct sign in that same interval the right-hand side of (11.12) would be negative and indicate that \( \Delta t \) is not minimized by \( y^* \). By assumption this is not possible so we conclude that \( G(x) = 0 \).

The change of the functional \( \Delta t \) to first order in the perturbation \( \delta y(x) \) is called its variational derivative. We will write our computation of this new type of derivative in the standard notation:

\[ \frac{\delta}{\delta y(x)} \Delta t[y] = G(x) = \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \]. \tag{11.13} \]

We think of \( \delta/\delta y(x) \) as giving the rate-of-change of the functional that follows it when \( y \) is varied only at the point \( x \). In this sense the variational derivative is like a partial derivative, where the infinite number of function values of \( y \) at the different points \( x \) are treated as independent variables.
The statement that all the variational derivatives of $\Delta t[y]$ vanish — for every $x$ — is a necessary condition for $\Delta t[y]$ to be minimized. At the particular function $y = y^*$ where this is supposed to be true, our derivation shows that

$$\left.\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\right|_{y^*} = 0.$$  (11.14)

We see that the form of the second-order differential equation satisfied by $y$, for the minimum-time curve $y^*$, has exactly the form of an Euler-Lagrange equation.