Lecture 11: February 22

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11.1 The calculus of variations

As physicists/mathematicians at the end of the 17^{th} century began exploring the analytical powers made possible by calculus, a particular problem was posed by Johann Bernoulli that would provide the foundation a century later for a new principle of mechanics. We will repeat this historical line of development, starting with Bernoulli's *brachistochrone problem* and the *calculus of variations* it inspired, then applying the result to mechanics where it is now known as *Hamilton's principle*.

11.1.1 The brachistochrone problem

Bernoulli asked for the curve $y(x)$ in a vertical plane along which a particle of mass m, acted on only by gravity (no friction), is transported in the shortest time between endpoints $(x = 0, y = 0)$ and $(x = l, y = 0)$, starting/arriving with zero velocity. Using elementary mechanics we can derive a formula for the time of transit T for a general curve $y(x)$ having the given endpoints.

By conservation of energy, the speed $v(y)$ of the mass satisfies

$$\frac{1}{2}mv^2 + mgy = 0, \tag{11.1}$$

with solution

$$v(y) = \sqrt{-2gy}. \tag{11.2}$$

The arc-length ds traversed by the mass along the curve during time dt is

$$ds = vdt, \tag{11.3}$$

and can be expressed as

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\, dx, \tag{11.4}$$

where y' is our shorthand for the derivative dy/dx. Solving for dt in (11.3) in terms of ds from (11.4) and v from (11.2), we obtain

$$\begin{aligned} T = \int dt \quad &= \quad \int_0^l \sqrt{\frac{1 + y'^2}{-2gy}}dx & (11.5) \\ &= \quad \int_0^l F(y, y', x)dx. & (11.6) \end{aligned}$$

The function F does not actually depend on x, but we include it in our general discussion since we later apply the principle to mechanics, where the Lagrangian $L(q, \dot{q}, t)$ takes the place of F.

11.1.2 The variational derivative

To solve the brachistochrone problem we want to minimize the elapsed time "functional",

$$T[y] = \int_0^l F(y, y', x)dx, \tag{11.7}$$

with respect to all possible smooth curves $y(x)$ that join the given endpoints. We refer to T as a functional (rather than a function) to emphasize the fact that its value depends on a function (rather than a finite number of variables). In the following discussion we will not make use of

$$F(y, y', x) = \sqrt{\frac{1 + y'^2}{-2gy}}, \tag{11.8}$$

other than the fact that F has the three arguments y, y', and x.

Suppose that $y^\star(x)$ is the function that minimizes T. If that is the case, then it should not be possible to slightly perturb $y^\star$ and thereby decrease T. With that in mind, let's calculate T when the argument of this functional is $y^\star(x) + \delta y(x)$, where $\delta y(x)$ is an infinitesimally small function:

$$T[y^\star + \delta y] = \int_0^l \left(F|_{y^\star} + \left.\frac{\partial F}{\partial y}\right|_{y^\star} \delta y + \left.\frac{\partial F}{\partial y'}\right|_{y^\star} \frac{d}{dx}\delta y + O(\delta y^2) \right) dx. \tag{11.9}$$

The notation $|_{y^\star}$ indicates that the preceding expression (F, $\partial F/\partial y$, etc.) is to be evaluated with the function $y^\star$. Integrating by parts the third term of (11.9),

$$\int_0^l \left(\left.\frac{\partial F}{\partial y'}\right|_{y^\star} \frac{d}{dx}\delta y \right) dx = -\int_0^l \left(\frac{d}{dx} \left.\left(\frac{\partial F}{\partial y'} \right)\right|_{y^\star} \delta y \right) dx + \left. \left(\left.\frac{\partial F}{\partial y'}\right|_{y^\star} \delta y \right) \right|_{x=0}^{x=l}. \tag{11.10}$$

The second term vanishes because the perturbation must vanish at the endpoints ($\delta y(0) = \delta y(l) = 0$) in order that the curve still joins the original points of the brachistochrone problem. Subtracting $T[y^\star]$ from (11.9) and using the result of the partial integration (11.10), we obtain

$$\begin{aligned} T[y^\star + \delta y] - T[y^\star] &= \int_0^l \left(\left(\left.\frac{\partial F}{\partial y}\right|_{y^\star} - \frac{d}{dx} \left.\left(\frac{\partial F}{\partial y'} \right)\right|_{y^\star} \right) \delta y + O(\delta y^2) \right) dx & (11.11) \\ &= \int_0^l \left(G(x)\delta y(x) + O(\delta y^2) \right) dx. & (11.12) \end{aligned}$$

Now, if the function $G(x)$ that multiplies the perturbation $\delta y(x)$ is non-zero in some small interval, then by making $\delta y(x)$ non-zero and of the correct sign in that same interval, the right-hand side of (11.12) would be negative and indicate that T is not minimized by $y^\star$. By assumption this is not possible so we conclude that $G(x) = 0$ for all x.

The change of the functional T to first order in the perturbation $\delta y(x)$ is called its *variational derivative*. Keeping with convention, the notation for this new kind of derivative is the following:

$$\frac{\delta}{\delta y(x)} T[y] = G(x) = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right). \tag{11.13}$$

We think of $\delta/\delta y(x)$ as giving the rate-of-change of the functional that follows it when y is varied only at the point x. In this sense the variational derivative is like a partial derivative, where the infinite number of function values of y at the different points x are treated as independent variables.

The statement that all the variational derivatives of $T[y]$ vanish — for every x — is a necessary condition for $T[y]$ to be minimized. At the particular function $y = y^\star$ where this is supposed to be true, our derivation shows that

$$\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\bigg|_{y^\star} = 0. \tag{11.14}$$

We see that the form of the second-order differential equation satisfied by y, for the minimum-time curve $y^\star$, has exactly the form of an Euler-Lagrange equation.