Physics 3318: Analytical Mechanics	**Spring 2017**

Lecture 10: February 17

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10.1 Lagrangian mechanics (continued)

10.1.1 Conjugate momentum

The partial derivative

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}, \tag{10.1}$$

which appears in the Euler-Lagrange equations and also the definition of the Hamiltonian, defines the *momentum conjugate* to the generalized coordinate q_k. When q_k is a Cartesian coordinate, or the angle of rotation about a fixed axis, the conjugate momentum reduces to the familiar forms of "momentum":

$$\mathcal{L} = T = \frac{1}{2}M\dot{x}^2, \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = M\dot{x} = p_x = \text{linear momentum.} \tag{10.2}$$

$$\mathcal{L} = T = \frac{1}{2}I\dot{\theta}^2, \qquad \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\dot{\theta} = p_\theta = \text{angular momentum.} \tag{10.3}$$

In general, p_k will not be a familiar type of momentum at all. Yet the property of it being conserved is completely general, and tied to a very specific property of the Langrangian:

Momentum conservation: When q_k is absent from $\mathcal{L}$, the conjugate momentum p_k is constant in time.

This fact follows directly from the Euler-Lagrange equation for q_k:

$$\dot{p}_k = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad \text{(when } q_k \text{ is absent).} \tag{10.4}$$

Knowing that a certain number of conjugate momenta are constant usually greatly simplifies the solution of the equations of motion. Systems of equations of the form

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \text{constant}, \tag{10.5}$$

are first-order-in-time differential equations and often are easily solved. However, as the next section will show, constraints involving time derivatives (generalized velocities) can present new challenges.

10.1.2 Holonomic and non-holonomic constraints

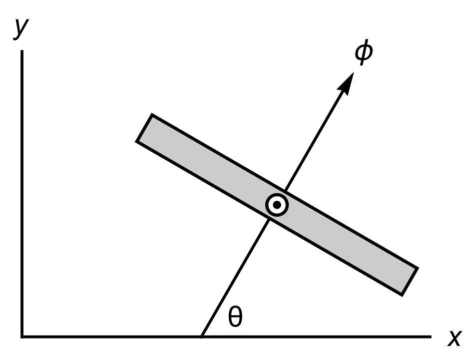
The systems we have considered up to now, where most generally the point-mass positions are expressed as

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_N; t), \tag{10.6}$$

are said to have *holonomic constraints*. To understand the need for this qualification — and the possibility of constrained motion that does not fit this form — we need an example.

Suppose we have a wheel constrained to always be upright on a flat surface, where it rolls without slipping. A reasonable physical realization would be a thin solid cylinder kept upright by a strong magnetic attraction to the surface. Let's count the number of degrees of freedom of this system.

- There are two degrees of freedom parameterized by the x and y coordinates of the wheel's center in the plane of the surface.
- The angle θ of the wheel's axis in the (x, y) plane is another degree of freedom.
- The rotation ϕ of the wheel about its axis is a fourth degree of freedom.



View from above, of a wheel rolling on the (x, y) plane. The wheel axis makes angle θ with respect to the x axis, and its arrowed-end represents the positive sense of rotation by ϕ. In addition to rolling by the ϕ rotation, the wheel may also pivot about the vertical axis through its center, thereby changing θ.

In tabulating the degrees of freedom we did not consider the rolling-without-slipping constraints:

$$v_x = 0, \qquad v_y = 0. \tag{10.7}$$

These are the components of the velocity of the point on the rim of the wheel that makes instantaneous contact with the surface. Since each constraint subtracts one degree of freedom, we end up with $4 - 2 = 2$ degrees of freedom. What should we use for our two generalized coordinates? Previously we were told that this is mostly a matter of convenience, since whichever two we choose should then determine the other two. So let's use θ and ϕ and try to determine x and y in terms of them.

The wheel center must move parallel to the plane of the wheel, otherwise the contact point on its rim will slip relative to the surface. The velocity of the contact point is a sum of two velocities: the velocity of the wheel center and the velocity of the rim relative to the center. Only the velocity $\dot{\phi}$ contributes to the latter, since the contact point is on the axis of the θ rotation. The velocity of the rim relative to the wheel center, along the line parallel to the wheel, is therefore $r\dot{\phi}$, where r is the radius of the wheel. Our two constraints (10.7) that the net velocity of the rim at the contact is zero thus take the following form:

$$\dot{x} = r\dot{\phi}\sin\theta \tag{10.8}$$
$$\dot{y} = -r\dot{\phi}\cos\theta. \tag{10.9}$$

We would like to integrate these equations in time — remembering that $\theta(t)$ is in general time dependent — and thereby arrive at equations

$$x = x(\theta, \phi), \qquad y = y(\theta, \phi). \tag{10.10}$$

But as we will see, this is impossible.

There are sequences of motions that demonstrate it is impossible for equations of the form (10.10) to exist. Consider motions that take the system from $(\theta, \phi) = (0, 0)$ to $(\theta, \phi) = (\pi, \phi_0)$. There are infinitely many ways of navigating between these points in the (θ, ϕ) plane; we will consider just two. In motion A we first rotate θ by π, and follow this by rotating ϕ through angle ϕ_0. As you might have anticipated, in motion B we perform the same rotations but in the reverse order.

Exercise: Make two plots, one showing the paths of motions A and B in the (θ, ϕ) plane, the other the corresponding paths of the wheel center in the (x, y) plane.

As this example shows, the (x, y) position of the wheel depends not just on the generalized coordinates (θ, ϕ), but also the *history* of the motion. Equations (10.10) cannot exist because they make no reference to history.

You may prefer a calculus-based argument that shows equations (10.10) cannot exist. Assuming the first equation exists, we could define a function

$$f(x, \theta, \phi) = x - x(\theta, \phi) \tag{10.11}$$

of three variables such that points on the surface

$$0 = f(x, \theta, \phi), \tag{10.12}$$

define x in terms of θ and ϕ. Taking the time derivative of this equation we obtain:

$$\begin{aligned} 0 &= \frac{df}{dt} && (10.13)\\ &= \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial \theta}\dot{\theta} + \frac{\partial f}{\partial \phi}\dot{\phi} && (10.14)\\ &= \dot{x} + \frac{\partial f}{\partial \theta}\dot{\theta} + \frac{\partial f}{\partial \phi}\dot{\phi}, && (10.15)\end{aligned}$$

since by (10.11) we know $\partial f/\partial x = 1$. We recognize (10.15) as the linear relationship among the three velocities, (10.8), and therefore

$$\frac{\partial f}{\partial \theta} = 0, \qquad \frac{\partial f}{\partial \phi} = -r\sin\theta. \tag{10.16}$$

But we now have a problem because

$$0 = \frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial \theta}\right) \neq \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \phi}\right) = -r\cos\theta. \tag{10.17}$$

More generally, a system of N degrees of freedom whose particle positions cannot be uniquely expressed in terms of N generalized coordinates is said to have *non-holonomic constraints.* The Lagrangian formulation of mechanics is incomplete until we can find a method for working with such constraints. We will see shortly that a new interpretation of the Euler-Lagrange equations, through a variational principle, provides such a method.