**Physics 3318: Analytical Mechanics** **Spring 2017**

# Lecture 1: January 27

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A natural reaction of physics students — from being on the receiving end of an often arcane selection of topics — is resignation. "Rotating reference frames" and "rigid body motion" are, it seems, just part of a bizarre initiation ritual. And while some of this may be true, it's time for you to prepare for the possibility that there might be a *fundamental* reason behind all of this.

The physicists' obsession with rotating things ("bodies") can be traced to a basic property of space: **rotational symmetry**. As an organism evolved to automatically comprehend a 3D world — a world that has rotational symmetry — you take this property of space for granted. But in mathematics, where things are examined with painstaking precision, there are other kinds of 3D worlds, with other symmetries. So look at the study of rigid body motion with the detachment of the mathematician, and be thankful that you did not end up in one of those truly strange worlds!

## 1.1 Mathematical description of rotated frames

To better articulate the consequences of a 3D world with rotational symmetry, we need some standard nomenclature and notation. We will have two frames of reference. The first is the **space frame** — an inertial frame — where the laws of motion take the form you learned in freshman mechanics. We will use the standard Cartesian coordinates $x$, $y$ and $z$ to describe position in the space frame.

As we will be interested in rigid bodies, our second frame of reference is the **body frame**. The "body" may be large (earth, mothership) and correspond to a particular non-inertial frame imposed by circumstances on the "observer". There are also many situations where an observer in the space frame tries to make sense of the tumbling motion of a rigid body (earth, football). In either case, we use a coordinate system *fixed in the body* for this frame. We will use coordinates $x'$, $y'$ and $z'$ for position in the body frame. By definition, every point fixed in the body (*e.g.* valve for inflating the football) has time-independent body-frame coordinates.

It is important to understand the distinction between coordinates and vectors when describing position. A position vector, written boldface, is an intrinsic property and independent of frame. For example, $\mathbf{r}$ might be the position of a particle. We can express $\mathbf{r}$ in terms of another set of vectors like this:

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}. \tag{1.1}$$

The (boldface) vectors on the right-hand-side are the *(unit) basis vectors of the space frame* and the scalar coefficients multiplying them are the coordinates of $\mathbf{r}$ in that frame. When the particle moves, only the coordinates change with time. The basis vectors of the space frame are constant:

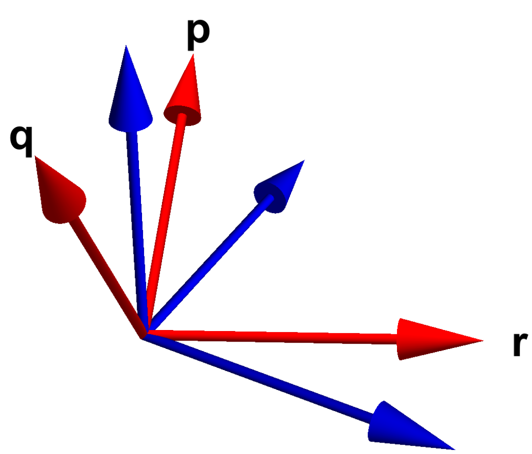
$$0 = \dot{\hat{\mathbf{x}}} = \dot{\hat{\mathbf{y}}} = \dot{\hat{\mathbf{z}}}. \tag{1.2}$$

We can also express $\mathbf{r}$ in terms of the *(unit) basis vectors of the body frame*:

$$\mathbf{r} = x'\,\hat{\mathbf{x}}' + y'\,\hat{\mathbf{y}}' + z'\,\hat{\mathbf{z}}'. \tag{1.3}$$

You should think of this always as applying at a particular instant of time. For example, the particle at **r** might be at rest, and yet the body frame basis vectors may be changing (for whatever reason), forcing the body frame coordinates to also change with time. This example underscores the fact that a triple of coordinates is not equivalent to a vector: the basis vectors associated with those coordinates are just as integral.

Below is a rendering of the space frame basis vectors in blue and the body frame basis vectors (for a particular rotation of the body) in red. The body basis vectors "point" to a trio of points **p**, **q** and **r**.




We can arrange the three coordinates of **p** in the two frames into columns and call them $p'$ (body frame) and $p$ (space frame), and similarly for the other two points:

$$p' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad q' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} \qquad r' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}. \tag{1.4}$$

The coordinates in the space frame are of course not arbitrary, but constrained by the geometrical properties $\mathbf{p}\cdot\mathbf{p} = 1$, $\mathbf{p}\cdot\mathbf{q} = 0$, etc. Combining all nine coordinates into the matrix

$$U = \begin{bmatrix} p_x & q_x & r_x \\ p_y & q_y & r_y \\ p_z & q_z & r_z \end{bmatrix}, \tag{1.5}$$

the full set of constraints is expressed compactly as

$$U^T U = \mathbb{1}, \tag{1.6}$$

where $U^T$ is the matrix transpose of $U$. The columns of $U$ are said to be "orthonormal".

Notice that

$$\begin{bmatrix} p_x & \cdots & \cdots \\ p_y & \cdots & \cdots \\ p_z & \cdots & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \tag{1.7}$$

is just the matrix equation $Up' = p$. The same matrix $U$ transforms *any* set of body-frame coordinates to the corresponding space-frame coordinates. We summarize this basic fact using the standard notation for a general position vector $\mathbf{r}$, whose coordinates are $r$ and $r'$ in the space and body frames:

$$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad r' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \qquad Ur' = r. \tag{1.8}$$

The orthogonal matrix $U$ is called the *rotation matrix*, or *instantaneous rotation matrix* because in general the body is not static but tumbling in space. In fact, all the complications we must deal with over the next several lectures are consequences of a time-dependent rotation matrix $U$.

## 1.2 Time-rate-of-change of the rotation matrix

Consider a point fixed in the body, so $\dot{r}' = 0$, but in a situation where $U$ changes with time. Now calculate the time-rate-of-change of the space frame coordinates — the velocity of the point:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = v = \dot{r} = \frac{d}{dt}(Ur') = \dot{U}r' \tag{1.9}$$

$$= \dot{U}(U^T U)r' \tag{1.10}$$

$$= \dot{U}U^T r. \tag{1.11}$$

Summarizing, we have the following general relationship between the coordinates and their time derivatives in the space frame:

$$v = \dot{r} = Ar, \qquad A = \dot{U}U^T. \tag{1.12}$$

**Exercise:** Show that the matrix $A$ is antisymmetric.

Since $A$ is antisymmetric, a valid but also nice way of naming the elements is the following:

$$A = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{1.13}$$

The rationale for this convention should be clear from the expression we get for the velocity:

$$v = Ar = \begin{bmatrix} \omega_y z - \omega_z y \\ \omega_z x - \omega_x z \\ \omega_x y - \omega_y x \end{bmatrix}. \tag{1.14}$$

These are the components of the vector $\boldsymbol{\omega} \times \mathbf{r}$. Since $v$ are the (space frame) components of the velocity vector $\mathbf{v}$, we've derived the freshman kinematic relationship

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \tag{1.15}$$

**Exercise:** Convey the content of equation (1.15) using your hand.

That the angular velocity vector $\boldsymbol{\omega}$ of freshman mechanics may be more usefully represented by the antisymmetric matrix $A$ is clear when we need to find the time-rate-of-change of the rotation matrix. Multiplying the definition of $A$ (1.12) on the right by $U$, we get

$$\dot{U} = AU. \tag{1.16}$$