Problem 1: Goldstein 8.7

We have:

Eq 45: \( H_1 = \frac{p^2}{2m} + \frac{k}{2}(x-v_0t)^2 \)

\[
\begin{align*}
\frac{\partial H_1}{\partial x} &= -\dot{p} = k(x - v_0t) \\
\frac{\partial H_1}{\partial p} &= \dot{x} = \frac{p}{m}
\end{align*}
\]

\[\Rightarrow \dot{x} = \frac{p}{m} = -\frac{k}{m}(x - v_0t) \quad (\text{**})
\]

Eq 8.47: \( H_2 = \frac{(p' - mv_0)^2}{2m} + \frac{k}{2}(x')^2 - \frac{mv_0^2}{2} \)

\[
\begin{align*}
\frac{\partial H_2}{\partial x'} &= -p' = kx' \\
\frac{\partial H_2}{\partial p'} &= \dot{x}' = \frac{(p' - mv_0)}{m}
\end{align*}
\]

\[\Rightarrow \ddot{x}' = \frac{1}{m} \frac{d}{dt} (p' - mv_0) = \frac{\dot{p}'}{m}
\]

\[\ddot{x}' = -\frac{k}{m} x'
\]

We have \( x' = x - v_0t \).

Therefore \( \ddot{x}' = \ddot{x} \)

\[\Rightarrow \ddot{x} = -\frac{k}{m} (x - v_0t) \quad (\text{**}) \quad (\text{**}) \quad (\text{**})
\]

(\text{**}) \& (\text{**}) \text{ match!}
Problem 2: Hypocycloid Hamiltonian

Recall from HW #5:

\[
\begin{align*}
    x = (R-r) \cos \theta + r \cos \left( \frac{R-r}{r} \theta \right) \\
y = (R-r) \sin \theta + r \sin \left( \frac{R-r}{r} \theta \right)
\end{align*}
\]

With \( R = 2r \), these become

\[
\begin{align*}
x &= 2r \cos \theta \\
y &= r \sin \theta - r \sin \theta = 0 \quad \text{The motion of the mass is simplified, depends only on } x
\end{align*}
\]

\[
\begin{align*}
    \dot{x} &= -2r \dot{\theta} \sin \theta \\
    \dot{y} &= 0
\end{align*}
\]

The Lagrangian is

\[
L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgx
\]

\[
= \frac{1}{2} m (-2r \dot{\theta} \sin \theta)^2 - 2mgr \cos \theta
\]

\[
L = 2mr \dot{\theta}^2 \sin^2 \theta - 2mgr \cos \theta
\]

The Hamiltonian is:

\[
H = \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L
\]

\[
= 4mr \dot{\theta}^2 \sin^2 \theta - 2mr \dot{\theta}^2 \sin^2 \theta + 2mgr \cos \theta
\]

\[
H = 2mr \dot{\theta}^2 \sin^2 \theta - 2mgr \cos \theta
\]

or

\[
H = \frac{P_{\theta}^2}{8mr^2 \sin^2 \theta} - 2mgr \cos \theta
\]

where \( P_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = 4mr^2 \dot{\theta} \sin^2 \theta \)

I set \( m = 1 \)

\( r = 1 \)

\( g = 10 \)

\( H \) is periodic with period \( 2\pi \) and is an even function of \( \theta \).

Therefore we are able to restrict the range of \( \theta \) between \(-\pi \) and \( \pi \).
Define \( k = \frac{p}{r} \).

- If \( k \in \mathbb{Z} \), the mass returns to its starting point. Specifically, if \( k = 2 \), the mass will return after a \( 2\pi \) revolution.

- If \( k \) is rational, the mass will also return to the starting point after a finite number of revolutions.

- If \( k \) is irrational, the curve traced by the point mass never closes (i.e., never returns to starting position).

When \( \theta = \pm \pi \), the topology of the phase space orbit changes.

\[
H^* = 0 - 2mgr \cos(\pm \pi) = 2mgr.
\]
Consider the following small rectangle in phase space:

\[
\begin{array}{c}
P
\end{array}
\]

Initial conditions at \( t = 0 \):

\[
\begin{align*}
q_0 &< q < q_0 + dq \\
p_0 &< p < p_0 + dp
\end{align*}
\]

At \( t = \varepsilon \):

\[
\begin{align*}
q_0 &\rightarrow q_0 + \varepsilon \frac{\partial H}{\partial p} \\
p_0 &\rightarrow p_0 + \varepsilon \frac{\partial H}{\partial q}
\end{align*}
\]

\[
\begin{align*}
q_0 + dq &\rightarrow (q_0 + dq) + \varepsilon \frac{\partial H}{\partial p} = q_0 + dq + \varepsilon \frac{\partial H}{\partial p} \\
p_0 + dp &\rightarrow (p_0 + dp) + \varepsilon \frac{\partial H}{\partial q} = (p_0 + dp) - \varepsilon \frac{\partial H}{\partial q}
\end{align*}
\]

It is clear that the initial conditions evolved into the image of the rectangle under the following transformation:

\[
\begin{align*}
q'(q,p) &= q + \varepsilon \frac{\partial H}{\partial p} \\
p'(q,p) &= p - \varepsilon \frac{\partial H}{\partial q}
\end{align*}
\]

Now, let's check that the transformation is locally area-preserving.

The Jacobian of the transformation is:

\[
J = \begin{bmatrix}
\frac{\partial q'}{\partial q} & \frac{\partial q'}{\partial p} \\
\frac{\partial p'}{\partial q} & \frac{\partial p'}{\partial p}
\end{bmatrix} = \begin{bmatrix}
1 + \varepsilon \frac{\partial H}{\partial p} & \varepsilon \frac{\partial^2 H}{\partial q \partial p} \\
\varepsilon \frac{\partial^2 H}{\partial q \partial p} & 1 - \varepsilon \frac{\partial H}{\partial q}
\end{bmatrix}
\]

The Jacobian determinant is

\[
\det(J) = (1 + \varepsilon \frac{\partial H}{\partial p})(1 - \varepsilon \frac{\partial H}{\partial q}) + \varepsilon^2 \frac{\partial^2 H}{\partial q \partial p} \cdot \frac{\partial^2 H}{\partial q^2} = 1 - \varepsilon^2 \frac{\partial^2 H}{\partial q \partial p} + \varepsilon^2 \frac{\partial^2 H}{\partial q^2} \cdot \frac{\partial^2 H}{\partial q^2} = 1 + O(\varepsilon^3) = 1
\]

to first order in \( \varepsilon \).
For finite times, area is also preserved because we can construct a finite time by compounding many infinitesimal transformations (each of which preserves area).

Extending the calculation to $N$ degrees of freedom:

\[
\begin{align*}
\begin{aligned}
\frac{2H}{\delta q_i} &= -\dot{p}_i \\
\frac{2H}{\delta p_i} &= q_i
\end{aligned}
\end{align*}
\]

After $t = \varepsilon$:

\[
\begin{align*}
\begin{aligned}
q_i' &= q_i + \varepsilon \frac{\partial H}{\partial p_i} \\
p_i' &= p_i - \varepsilon \frac{\partial H}{\partial q_i}
\end{aligned}
\end{align*}
\]

The Jacobian is:

\[
J = \begin{bmatrix}
\frac{\partial q_1}{\partial q_1} & \frac{\partial q_1}{\partial p_1} & \cdots & \frac{\partial q_1}{\partial q_N} & \frac{\partial q_1}{\partial p_N} \\
\frac{\partial q_2}{\partial q_1} & \frac{\partial q_2}{\partial p_1} & \cdots & \frac{\partial q_2}{\partial q_N} & \frac{\partial q_2}{\partial p_N} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial q_N}{\partial q_1} & \frac{\partial q_N}{\partial p_1} & \cdots & \frac{\partial q_N}{\partial q_N} & \frac{\partial q_N}{\partial p_N}
\end{bmatrix}
\]

All of these entries are zero to 1st order in $\varepsilon$ except for those that give rise to a block diagonal matrix (each block is a $2 \times 2$ matrix) from before:

\[
J = \begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_N
\end{bmatrix}
\]

where $A_i = \begin{bmatrix}
\frac{\partial q_i}{\partial q_i} & \frac{\partial q_i}{\partial p_i} \\
\frac{\partial q_i}{\partial p_i} & \frac{\partial p_i}{\partial p_i}
\end{bmatrix}$ and $\det A_i = 1 + O(\varepsilon^2)$

\[
\det J = \det A_1 \times \det A_2 \times \cdots \times \det A_N
\]

\[
= (1 + O(\varepsilon^2)) \times \cdots \times (1 + O(\varepsilon^2))
\]

\[
= [1 + O(\varepsilon^2)]^N = 1 \quad \text{to first order}
\]