Problem 1: Goldstein 1.8

Let \( L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt} \)

The action is given by:

\[
S'[L'] = \int_{t_1}^{t_2} (L + \frac{dF}{dt}) \, dt
\]

\[
S'[L'] = \int_{t_1}^{t_2} L \, dt + \int_{t_1}^{t_2} \frac{dF}{dt} \, dt
\]

\[
S'[L'] = S[L] + \left( F(q(t_2), t_2) - F(q(t_1), t_1) \right)
\]

The equations of motion are obtained by making the action extremal. The all-important constraint is that we make the endpoints \( q(t_1), q(t_2) \) fixed.

\[
\delta S = \delta S + \delta \left( F(q(t_2), t_2) - F(q(t_1), t_1) \right)
\]

\[
= 0 \quad \text{since endpoints are fixed}
\]

\[
\delta S[L'] = \delta S[L]
\]

\( \Rightarrow \) The equations of motion are invariant
Problem 2: Brachistochrone

\[ T[y(x)] = \int_0^L \sqrt{\frac{1 + \left( \frac{dy}{dx} \right)^2}{2gy}} \, dx = \int_0^L \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} \, dx = \int_0^L F(y(x), x) \, dx \]

a) Euler-Lagrange equation

We want \( S[T[y(x)] = 0 \)

\[ \int_0^L \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) y' \, dx = 0 \]

\[ \Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \]

We have:

\[ \frac{\partial F}{\partial y} = \frac{g \sqrt{1 + y'^2}}{(-2gy)^{3/2}} \]

\[ \frac{\partial F}{\partial y'} = \frac{y'}{\left[2gy(1+y'^2)\right]^{1/2}} \]

b) Parametrization

\[ \begin{cases} x = R\theta - R\sin \theta \\ y = -R + R\cos \theta \end{cases} \]

\[ y' = \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = -\frac{R \sin \theta}{-R \cos \theta + R} = \frac{\sin \theta}{\cos \theta - 1} \]

\[ \Rightarrow \frac{\partial F}{\partial y} = \frac{g \left[\frac{\sin \theta}{(\cos \theta - 1)^{3/2}}\right]}{\left[\frac{1 - \cos \theta}{(1 - \cos \theta)^2}\right]^{3/2}} = \frac{1}{2\sqrt{gR}} \cdot \frac{1}{\left(\frac{1 - \cos \theta}{(1 - \cos \theta)^2}\right)^{1/2}} \]

\[ \Rightarrow \frac{\partial F}{\partial y'} = \frac{\sin \theta / (\cos \theta - 1)}{\left[2gR \left(1 - \cos \theta\right)^{2/3}\right]^{1/2}} = \frac{1}{2\sqrt{gR}} \cdot \frac{\sin \theta}{\cos \theta - 1} \]
\[ \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = \left( \frac{d}{dx} \right) \left( \frac{\partial F}{\partial y'} \right) \]

\[ = \frac{1}{R(1-\cos\Theta)} \cdot \frac{d}{d\Theta} \left( \frac{\sin\Theta}{\cos\Theta-1} \right) \cdot \frac{1}{2\sqrt{gR}} \]

\[ \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = \frac{1}{2\sqrt{gR^2}} \cdot \frac{1-\cos\Theta}{1-\cos\Theta} \cdot \frac{1}{(1-\cos\Theta)^2} \]

Therefore \[ \frac{\partial F}{\partial y} = \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \]

ie the parametrization satisfies E-L equation

c) Curve

\[ L = 2\pi R \]

d) Time of transit

\[ T[y(x)] = \int_0^L \left[ \frac{1 + y'^2}{-2g y} \right]^{\frac{1}{2}} dx = \int_0^{2\pi} \left( \frac{2(1-\cos\Theta)}{2gR(1-\cos\Theta)} \right)^{\frac{1}{2}} R(\cos\Theta + 1) \, d\Theta \]

\[ = \int_0^{2\pi} \left( \frac{1}{Rg(1-\cos\Theta)^2} \right)^{\frac{1}{2}} R(1-\cos\Theta) \, d\Theta \]

\[ = \int_0^{2\pi} \sqrt{\frac{R}{g}} \, d\Theta = 2\pi \sqrt{\frac{R}{g}} = \sqrt{\frac{2\pi L}{g}} \]
In class, we found that the best transit time for a rectangular plane track is \( T_0 = \sqrt{\frac{8L}{g}} \).

Therefore \( T = \sqrt{\frac{2\pi L}{g}} \) < \( T_0 = \sqrt{\frac{8L}{g}} \).
Problem 3: Brachistochrone Jr

We know that the track must be the form of a cycloid. But first, we need to argue that the slope of the path at the wall \((x=0)\) must be zero.

Assume a non-zero slope at the wall. If we consider a similar path except that we replace the end near the wall with a horizontal segment (i.e., make a small deviation), we find that variation in \(T\) is first order. This is because the slope (first order) is non-zero! We conclude that the slope at the wall must be zero if we were to have a stationary (extremal) track.

There is a reflection symmetry about the wall. Under this symmetry operation, the equations of motion are to remain invariant. Therefore, the minimum time trajectory from A to O is the same as from B to O.

Additionally, we have time reversal symmetry. This implies that the time from B to O is the same as from O to B.

Thus:

\[
T = T_{A\to O} + T_{O\to B} = T_{A\to O} + T_{B\to O} = T_1 + T_2 = 2T_1
\]

\[
\Rightarrow T_1 = \frac{T}{2} = \pi \sqrt{\frac{R}{g}}
\]
Problem 4: Extrema of the harmonic oscillator Action

\[ S [x(t)] = \int_0^T \left( \frac{1}{2} \dot{x}^2 - \frac{1}{2} m \omega_0^2 x^2 \right) \, dt \]

\[ x(0) = x_1 \quad \& \quad x(T) = x_2 \quad \text{are fixed endpoints} \]

Let: \( x(t) = \ddot{x}(t) + Sx(t) \)

where \( \ddot{x}(t) \) satisfies E-L equations

\( Sx(t) \) is not small

(a) Show \( S[x(t)] \leq S[\ddot{x}(t)] + SS \)

\[ S[x(t)] = S[\ddot{x}(t) + Sx(t)] \]

\[ = \int_0^T \left[ \frac{1}{2} m (\dddot{x} + S\dot{x} + S^2)^2 - \frac{1}{2} m \omega_0^2 (\dddot{x} + S\dot{x} + S^2)^2 \right] \, dt \]

\[ = \int_0^T \left( \frac{1}{2} m \dddot{x}^2 - \frac{1}{2} m \omega_0^2 \dddot{x}^2 \right) \, dt + \int_0^T \left( \frac{1}{2} m (S\dot{x})^2 - \frac{1}{2} m \omega_0^2 (S\dot{x})^2 \right) \, dt \]

\[ + \int_0^T \left[ \frac{1}{2} m 2 \dot{x} \dddot{x} \dot{x} - \frac{1}{2} m \omega_0^2 2 \dot{x} \dddot{x} \dot{x} \right] \, dt \]

\[ = S[\ddot{x}(t)] + SS + \int_0^T \left( m \dot{x} \dddot{x} - m \omega_0^2 \dddot{x} \dddot{x} \right) \, dt \]

Last term: \[ = m \int_0^T (\dddot{x} \dddot{x} - m \omega_0^2 \dddot{x} \dddot{x}) \, dt \]

\[ = m \int_0^T \dddot{x} \dddot{x} - m \omega_0^2 \int_0^T \dddot{x} \dddot{x} \, dt \]

\[ = m \left( \dddot{x} \dddot{x} \bigg|_0^T - m \int_0^T \dddot{x} \dddot{x} \, dt - m \omega_0^2 \int_0^T \dddot{x} \dddot{x} \, dt \right) \]

\[ = 0 - m \int_0^T \dddot{x} \dddot{x} (\dddot{x} - m \omega_0^2 \dddot{x}) \, dt \]
For a harmonic oscillator, if \( x \) is the extremal trajectory, then the equation of motion is
\[
m \ddot{x} = mw_0^2 \ddot{x}
\]

\( \Rightarrow \) Last term = 0

Therefore:
\[
S[x(t)] = S[x(\dot{x})] + \delta S
\]

b) Consider \( \delta x(\dot{t}) = \Delta \sin (\frac{N\pi t}{T}) \):

\[
\delta x = \frac{N\pi}{T} \Delta \cos \left( \frac{N\pi t}{T} \right)
\]

\[
\delta S = \int_0^T \left[ \frac{1}{2} m \left( \frac{N\pi}{T} \Delta \cos \left( \frac{N\pi t}{T} \right) \right)^2 - \frac{1}{2} mw_0^2 \left( \Delta \sin \frac{N\pi t}{T} \right)^2 \right] dt
\]

\[
\delta S = \frac{1}{2} m \Delta^2 \frac{1}{T^2} \int_0^T \left[ N^2 \pi^2 \cos^2 \left( \frac{N\pi t}{T} \right) - w_0^2 T^2 \sin^2 \left( \frac{N\pi t}{T} \right) \right] dt
\]

\[
\delta S = \frac{m \Delta^2}{2T^2} \frac{T}{2} \left[ N^2 \pi^2 - w_0^2 T^2 \right]
\]

\[
\delta S = \frac{m}{4T^2} \left( N^2 \pi^2 - w_0^2 T^2 \right) \Delta^2
\]

\[
\delta S = c_N \Delta^2
\]

\( (w_0 T)^2 > (N\pi)^2 \Rightarrow c_N < 0 \Rightarrow \delta S < 0 \)

\( (w_0 T)^2 < (N\pi)^2 \Rightarrow c_N > 0 \Rightarrow \delta S > 0 \)