Assignment 4 solutions

Goldstein problem

Let 1̂, 2̂ and 3̂ be the three principal-moment body axes of your Goldstein textbook. When Goldstein is freely tumbling through space these axes are time dependent. At $t = 0$ the axes are aligned with the space axes:

$$
1̂(0) = \hat{x} \quad 2̂(0) = \hat{y} \quad 3̂(0) = \hat{z}.
$$

The angular velocity of Goldstein at $t = 0$,

$$\omega(0) = \omega_1(0)1̂(0) + \omega_2(0)2̂(0) + \omega_3(0)3̂(0),$$

is specified by the three initial angular velocity components $\omega_1(0)$, $\omega_2(0)$ and $\omega_3(0)$. From this information, how would one calculate the axes of Goldstein at later times?

Write down the loop of a computer program that in one execution computes the axes and the component angular velocities at time $t + dt$ starting with the values of these variables at time $t$. Your program will need the principal moments of inertia $I_1$, $I_2$ and $I_3$ of Goldstein, all of which are different.

Note: Your answer may be in schematic form; you are not expected to run the program on a computer.

Solution

The Euler equations will tell us how $\omega_1$, $\omega_2$ and $\omega_3$ change with time, but that will not give us the angular velocity vector $\omega$ since the basis vectors 1̂, 2̂ and 3̂ are themselves changing with time. Fortunately, by being fixed in the body, the change in these vectors is given by the precession equations:

$$\dot{1̂} = \omega \times 1̂ = \omega_3 2̂ - \omega_2 3̂$$

and similarly for the other two basis vectors.

Our time-stepping loop will therefore update the three scalars $\omega_1$, $\omega_2$ and $\omega_3$ and the three vectors 1̂, 2̂ and 3̂. The next page shows the initialization and loop in pseudocode. The $:=$ symbol is the assignment operator.
\[ \hat{1} := \dot{x} ; \]
\[ \hat{2} := \dot{y} ; \]
\[ \hat{3} := \dot{z} ; \]
\[ \omega_1 := \omega_1(0) ; \]
\[ \omega_2 := \omega_2(0) ; \]
\[ \omega_3 := \omega_3(0) ; \]

for \( t = 0 ; t < T ; t = t + dt \) \{ \}
\[ \omega_1 := \omega_1 + dt \left( \frac{I_2 - I_1}{I_1} \right) \omega_2 \omega_3 ; \]
\[ \omega_2 := \omega_2 + dt \left( \frac{I_3 - I_1}{I_2} \right) \omega_3 \omega_1 ; \]
\[ \omega_3 := \omega_3 + dt \left( \frac{I_1 - I_2}{I_3} \right) \omega_1 \omega_2 ; \]
\[ \hat{1} := \hat{1} + dt \left( \omega_3 \hat{2} - \omega_2 \hat{3} \right) ; \]
\[ \hat{2} := \hat{2} + dt \left( \omega_1 \hat{3} - \omega_3 \hat{1} \right) ; \]
\[ \hat{3} := \hat{3} + dt \left( \omega_2 \hat{1} - \omega_1 \hat{2} \right) ; \]
\}