Assignment 8

Due date: Wednesday, March 26

Goldstein problems

2.18 Use generalized coordinates (no Lagrange multipliers) on this problem.

Polymer model

Derive the equations of motion for the polymer chain model described in class, but for the case of $N = 3$ mass points. Use the method of Lagrange multipliers and, by solving for them explicitly, eliminate them from the equations of motion. **You do not have to solve the equations of motion.**

For the $N = 2$ case analyzed in lecture, the single Lagrange multiplier turned out to be constant in time. Is this also true for the two Lagrange multipliers in this problem?

Although solving for the $\lambda$’s will seem like a tedious exercise, and increasingly cumbersome for $N > 3$, the fact remains that these equations are linear in the $\lambda$’s and lend themselves to efficient numerical (as opposed to algebraic) methods. In a large scale simulation, say with $N = 1000$, the computer code would use a linear equation solver to numerically calculate the $\lambda$’s in each time step.

Motion with a vector field constraint

A point mass $m$ is constrained to move in the plane so that its velocity is always perpendicular to a (static) vector field:

$$\mathbf{c}(x, y) \cdot \mathbf{v} = 0,$$

$$\mathbf{c}(x, y) = c_x(x, y) \hat{x} + c_y(x, y) \hat{y}.$$  

In variational language, this means that the variations at time $t$, $\delta x(t)$ and $\delta y(t)$, where we hold $x(t \pm \Delta t)$ and $y(t \pm \Delta t)$ fixed, satisfy the linear constraint

$$c_x \delta x(t) + c_y \delta y(t) = 0.$$

This is a non-holonomic constraint and the Euler-Lagrange equations take the form (as in the rolling disk system from lecture):

$$\frac{\delta S}{\delta x(t)} = \lambda(t)c_x$$

$$\frac{\delta S}{\delta y(t)} = \lambda(t)c_y,$$
where $S$ is the action without the constraint (time integral of the kinetic energy of a mass moving in the plane).

Your assignment is to derive equations of motion for $x(t)$ and $y(t)$ that do not involve the unknown Lagrange multiplier $\lambda(t)$. Start by writing out (2) and (3), and substituting $\ddot{x}$ and $\ddot{y}$ from these into the expression you get by taking one time derivative of (1). This will allow you to solve for $\lambda(t)$, which you can then substitute into (2) and (3).

**Time-translation symmetry**

Consider a Lagrangian with no direct time dependence:

$$L = L(q_1, \ldots, q_N; \dot{q}_1, \ldots, \dot{q}_N).$$

Let $L'$ be the Lagrangian where the following continuous transformation, parameterized by $s$, is applied to the coordinates:

$$q_i(t) \rightarrow Q_i(t, s) = q_i(t + s) \quad i = 1, \ldots, N.$$  \hfill (4)

Show that

$$\left. \frac{dL'}{ds} \right|_{s=0} = \frac{dF}{dt}$$  \hfill (5)

for some function $F$; find $F$.

By Noether’s theorem, Lagrangians with property (5) automatically lead to the conserved quantity

$$I = \sum_{i=1}^{N} p_i \left. \frac{dQ_i}{ds} \right|_{s=0} - F.$$  

For the transformation (4) and the corresponding $F$ you found above, the quantity $I$ has another name: what is it?