Assignment 7

Due date: Wednesday, March 19

Goldstein problems

1.8 Redo this problem by first finding a very simple relationship between the corresponding actions. Then, by making use of the all-important constraints on the allowed variations, argue that an extremal action solution for one system will also be an extremal action solution for the other.

Brachistochrone

Starting from the time-functional for the Brachistochrone problem

\[ \tau[y(x)] = \int_0^L \sqrt{\frac{1 + (dy/dx)^2}{-2gy}} dx, \]

derive the Euler-Lagrange equation for the minimum-time curve \( y(x) \).

Next, show that your equation is satisfied by the following parametric representation of the curve,

\[
x = R \theta - R \sin \theta \\
y = -R + R \cos \theta,
\]

where \( R \) is a parameter. Sketch this curve as \( \theta \) ranges from 0 to 2\( \pi \). What should be the value of \( R \) so the curve spans a distance \( L \) between the two points where it intercepts the horizontal plane (\( y = 0 \))?

Finally, plug your parametric solution into the time-functional, integrate, and obtain the time of transit. Compare your answer with the bound on the transit time we found in the first lecture.

Brachistochrone Jr.

Consider this variation on the Brachistochrone problem:

A particle starts at a distance \( L \) from a vertical wall. What path should it take, when acted upon by gravity, so that it arrives at the wall in the minimum time?

Solve this problem with a symmetry argument. This is a technique not normally taught in textbooks, as there are no general recipes. What you are trying to do in this case is to relate the wall-version of the Brachistochrone problem to the standard
version without the wall (and thereby avoid having to write down lots of equations). This is a fun problem to discuss with others. The challenge is not just to get the right answer (for the transit time), but a tight argument.

**Extrema of the harmonic oscillator action**

The action functional for a 1D harmonic oscillator is

$$S[x(t)] = \int_0^T \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m\omega_0^2 x^2 \right) dt,$$

and the trajectory endpoints are fixed as

$$x(0) = x_1, \quad x(T) = x_2.$$

In this problem you will study arbitrary trajectories when expressed in the form

$$x(t) = \tilde{x}(t) + \delta x(t),$$

where $\tilde{x}(t)$ is an extremal trajectory given by Hamilton’s principle (and therefore satisfies the Euler-Lagrange equation) and $\delta x(t)$ is whatever is left over and is not assumed to be small.

(a) Show that $S[x(t)] = S[\tilde{x}(t)] + \delta S$, where

$$\delta S = \int_0^T \left( \frac{1}{2} m \dot{\delta x}^2 - \frac{1}{2} m\omega_0^2 \delta x^2 \right) dt.$$

Since $\delta x(0) = \delta x(T) = 0$, consider perturbations having the form

$$\delta x(t) = \Delta \sin \left( N \pi t / T \right),$$

where $\Delta$ is the amplitude of the perturbation and the integer $N$ counts the number of wiggles between the endpoints.

(b) Using the above form for $\delta x(t)$, show that $\delta S = c_N \Delta^2$, and determine the constant $c_N$. Further, show that for the case $\omega_0 T > \pi$ (trajectories that span more than one half-period), $c_N$ can have either sign, depending on $N$. The action functional thus will not always be a simple minimum or maximum at the extremum, but more generally, a “saddle” having both signs of curvature, depending on the “direction” in the space of perturbations.