Homework 3 solutions

1. (a) When we use the term “average size” we generally assume that both the set of objects being characterized, and the definition of size for individual objects, are obvious for our intended audience. In the case of the function $c$ we can simply say that this is the average size of clusters (connected subgraphs) — nothing more needs to be said. For the function $s$ we need to be more careful. First, the set of objects is not the set of clusters but the sets of “vertices connected to particular vertices”. More colloquially, call the sets “followers”, “disciples”, etc. to make it clear they pay homage to a single vertex. The average size is now the average number of vertices in each “cult” (including the enlightened leader).

(b) By differentiating the Taylor series for $v(z)$ we find $v' = w/z$, and $v' = e^{w}$ using the implicit equation for $w$. Differentiating the implicit equation with respect to $z$ we obtain $1 = (1 - w)e^{-w}w'$. Solving this for $e^{w}$ and substituting into the equation for $v'$ we get the promised differential equation. The solution of the differential equation is $v = w - w^2/2$, where the issue of the constant is settled by inspecting the two Taylor series.

(c) From lecture we know $f = w/d$. Also, the formula for $f$ in terms of cluster numbers $n_k$ is to $\sum_{k=1}^{\infty} \langle n_k \rangle / n$ (denominator of $c$) just as the Taylor series for $w$ is to the Taylor series for $v$ (an extra factor of $k$). Thus

$$c = \frac{w}{v} = \frac{1}{1 - w/2} = \frac{1}{1 - fd/2},$$

and the two forms given in the problem follow from the fact that $f = 1$ for $d < 1$ and (from lecture) $f = \bar{d}(d)/d$ for $d > 1$. Unlike $s(d)$, $c(d)$ does not diverge at $d = 1$ but has a cusp there with maximum value $c = 2$.

2. (a) Each Gaussian integral produces the factor $(\pi/\alpha)^{(D/2)(k+l-1)}$ and combines with $\rho^{k+l-1}$ to give $d^{k+l-1}$. Following the hint, almost all the Gaussian integrals are paired with integrals of the opposite sign and equal value because $\tau(C)^{D/2} = 1$ is independent of $C$ when $D = 0$. In particular, each of the factors $P_{112}(C)$ always produces such a pair (of oppositely signed integrals) when $C$ is not a complete graph. Also, in each of the factors $f(z_j)$ all the integrals are paired expect for one, which is the final $-1$. Summarizing, the expression above for $\langle n_k \rangle / n$ collapses to a sum over just the complete graphs $C$ and the single unpaired integral in each term contributes $(-1)^l$:

$$\langle n_k \rangle / n = \sum_{l=0}^{\infty} \frac{d^{k+l-1}}{k! l!} (-1)^l = \frac{d^{k-1}}{k!} e^{-d}.$$
(b) From the explicit formula for $\langle n_k \rangle / n$ it is easy to perform the sums on $k$ to get the three statistics functions for percolation at $D = 0$:

$$f = \sum_{k=1}^{\infty} k \langle n_k \rangle / n = e^{-d} \sum_{k=1}^{\infty} \frac{d^{k-1}}{(k-1)!} = 1.$$  

$$s = \sum_{k=1}^{\infty} k^2 \langle n_k \rangle / n = e^{-d} \frac{d}{dd} \left( \sum_{k=1}^{\infty} \frac{d^k}{(k-1)!} \right) = e^{-d}(1 + d)e^d = 1 + d.$$  

$$\sum_{k=1}^{\infty} \langle n_k \rangle / n = e^{-d} \sum_{k=1}^{\infty} \frac{d^{k-1}}{k!} = e^{-d}(e^d - 1)/d = (1 - e^{-d})/d.$$  

$$c = \frac{f}{\sum_{k=1}^{\infty} \langle n_k \rangle / n} = \frac{d}{1 - e^{-d}}.$$

I’m sure it was a shock for you to learn that the average number of vertices per cluster diverges, in zero dimensions, when the average degree equals a nonzero multiple of $2\pi i$. More seriously, the $D = 0$ limit is a useful “sum rule” for checking the tabulation of terms in nonzero $D$. This is demonstrated in the next problem.

3. In lecture we obtained the low density (low average degree) expansion of the $s$ statistic:

$$s = \sum_{G} \Sigma(2, G) \tau(G)^{-D/2} \frac{d^{v(G)-1}}{\sigma(G)},$$

where the sum is over connected, unlabeled graphs $G$, $v(G)$ is the number of vertices in $G$, $\sigma(G)$ is the order of the symmetry group of $G$, $\tau(G)$ is the number of spanning trees in $G$, and $\Sigma(2, G)$ is the (still nameless) property of $G$ given by the following formula:

$$\Sigma(2, G) = (-1)^{e(G)} \sum_{\text{cores } C \in G} v(C)^2 (-1)^{e(C)},$$

where $e(\cdot \cdot \cdot)$ counts the number of edges and a connected “core” subgraph $C$ is defined by the property that eliminating $C$ from $G$ leaves behind only isolated vertices.

The six connected graphs on four vertices have self-explanatory names “mercedes”, “diamond”, “path”, “martini”, “square”, and “complete”. Their properties are tabulated below:

In problem 2 we found that all the $s$ series coefficients beyond $d^3$ vanish for $D = 0$. This gives us a check on the $d^3$ coefficient:

$$\sum_{G} \frac{\Sigma(2, G)}{\sigma(G)} = \frac{0}{6} + \frac{14}{4} + \frac{2}{2} + \frac{-4}{2} + \frac{-12}{8} + \frac{-24}{24} = 0.$$
4. (a) We need to recalculate the contribution of a $k$-cycle when there is a harmonic potential:

$$z_k = \int d^3 r_1 \cdots d^3 r_k \langle r_2 r_3 \cdots r_1 | e^{-\beta H} | r_1 r_2 \cdots r_k \rangle$$

$$= \int d^3 r_1 \cdots d^3 r_k \langle r_2 | e^{-\beta H_1} | r_1 \rangle \langle r_3 | e^{-\beta H_1} | r_2 \rangle \cdots \langle r_1 | e^{-\beta H_1} | r_k \rangle$$

$$= \int d^3 r_1 \langle r_1 | e^{-k \beta H_1} | r_1 \rangle$$

$$= \text{Tr} \ e^{-k \beta H_1}.$$

Here we rewrote the matrix element of the multi-particle Hamiltonian $H$ in terms of a product of $k$ matrix elements of the single-particle Hamiltonian $H_1$. This is possible because in the absence of interactions $H$ is just a sum of single-particle Hamiltonians. Finally, we interpreted the resulting expression in terms of just one particle, where the integrals over $r_2, \ldots, r_k$ are sums over complete bases of intermediate states. To calculate the trace in the final line we switch to the basis of energy eigenstates:

$$\text{Tr} \ e^{-k \beta H_1} = \left( \sum_{n=0}^{\infty} e^{-k \beta \hbar \omega (n+1/2)} \right)^3$$

$$= \left( \frac{e^{-k \beta \hbar \omega /2}}{1 - e^{-k \beta \hbar \omega}} \right)^3$$

$$= (2 \sinh (k \beta \hbar \omega /2))^{-3}.$$

Following the suggestion (to be verified below) that $\beta_c$ scales as $1/N^{1/3}$, we simplify this expression for $\beta = \bar{\beta}/N^{1/3}$ in the limit of large $N$ and fixed $\bar{\beta}$:

$$z_k = \text{Tr} \ e^{-k \beta H_1} \approx \frac{N}{(k \beta \hbar \omega)^3}.$$
Using this $z_k$ in the grand canonical partition function (as in lecture) we sum over all combinations of cycle numbers $n_k$:

$$Z = \prod_{k=1}^{\infty} \sum_{n_k=0}^{\infty} \frac{(z_k e^{-k\mu})^{n_k}}{n_k! k^{n_k}}$$

$$\log Z = \sum_{k=1}^{\infty} z_k e^{-k\mu}/k = \frac{N}{(\beta\hbar\omega)^{3}} \sum_{k=1}^{\infty} \frac{e^{-k\mu}}{k^3}.$$ 

The chemical potential $\mu$ is determined by

$$N = -\frac{\partial}{\partial \mu} \log Z = \frac{N}{(\beta\hbar\omega)^{3}} \sum_{k=1}^{\infty} \frac{e^{-k\mu}}{k^3}$$

or the simple result

$$\left(\beta\hbar\omega\right)^{3} = \sum_{k=1}^{\infty} \frac{e^{-k\mu}}{k^3} \leq \zeta(3) \approx 1.202.$$  

Assuming the inequality is satisfied, and there is a solution for $\mu$, we examine (1) and notice that the $n_k$ have Poisson distributions with macroscopically large means

$$\langle n_k \rangle = z_k e^{-k\mu}/k = \frac{N}{(\beta\hbar\omega)^{3}} \frac{e^{-k\mu}}{k^4}.$$ 

We see that the decay in cycle numbers with cycle length $k$ is exponential for positive $\mu$, as it was for untrapped bosons, but is more rapid, $1/k^4$ vs. $1/k^{5/2}$, when $\mu = 0$ in the condensed phase.

(b) The fact that equation (2) determines the transition to $\mu = 0$ at a finite $\tilde{\beta}$ in the large $N$ limit confirms that we chose the correct scaling of transition temperature in the first part of this problem. The transition temperature is given by

$$T_c = \left(\frac{N}{\zeta(3)}\right)^{1/3} \hbar\omega \approx 0.94 N^{1/3} \hbar\omega.$$ 

(c) By the equipartition theorem, a classical particle in the harmonic potential has root-mean-square displacement of order $R \sim \sqrt{T}$ from the center of the potential. In a gas of $N$ such particles the mean interparticle separation is therefore $r \sim (R^3/N)^{1/3} \sim \sqrt{T}N^{-1/3}$. Particle indistinguishability manifests itself (and there is BEC) when the thermal wavelength $\lambda \sim 1/\sqrt{T}$ is comparable to $r$. Comparing the two lengths $r$ and $\lambda$ we see that this occurs when $T$ is of order $N^{1/3}$.