Homework 2 solutions

1. (a) Change to variables $y = Bx$:

$$\int d^N x \exp(-x^T Ax) = \det (B^{-1}) \int d^N y \exp(-y^T y)$$

and the formula follows because $\det A = (\det B)^2$ and each of the $N$ Gaussian integrals equals $\sqrt{\pi}$.

(b) For every eigenvector $v = [v_1, \ldots, v_{n-1}]$ of $\Delta_1$ with eigenvalue $\lambda$ we have $D$ eigenvectors of $A$ obtained by expanding each component of $v$ into a $D$-tuple with a single non-zero element in one of $D$ positions: $v_1 \to [0, \ldots, v_1, \ldots, 0]$. Since each of these $D$ eigenvectors of $A$ has eigenvalue $\lambda$, the determinant of $A$ is just the $D^{th}$ power of the determinant of $\Delta_1$.

(c) The sum in the exponential clearly has the structure of part (b): $D \times D$ identity blocks multiplied by the entries of an $n \times n$ matrix $\Delta(C)$. Call this $A_n$. Each edge $(i,j)$ of $C$ contributes, via $(x_i - x_j)^2$, $+1$ to the $(i,i)$ and $(j,j)$ diagonal elements and $-1$ to each of the off-diagonal elements $(i,j)$ and $(j,i)$. This is precisely the definition of the Laplacian of the graph $C$. Setting $x_1 = 0$ in $x^T A_n x$ is equivalent to the quadratic form $A$ obtained by block-expanding $\Delta_1(C)$. The formula then follows directly from parts (a) and (b).

(d) A tree of $n > 1$ vertices always has a “leaf”, that is a vertex of degree one. Let this be vertex $n$. Only one term $(x_n - x_i)^2$ (for some $i$) in the integrand involves $x_n$ and the integral over $x_n$ is just a simple Gaussian integral with value $\pi^{D/2}$. After this integral is performed the form of the remaining integral is again of the same form, but the tree graph has one less vertex. Since we do not integrate over the last vertex ($x_1$), the value of $I(C)$ is just $n - 1$ factors of $\pi^{D/2}$. Consistency with the formula of part (c) then implies $\det \Delta_1(C) = 1$.

2. (a) The key step is to observe the relationship that is generated when some edge $e$ of $C$ is removed or contracted, forming two other graphs. Contraction of edge (12) is illustrated on the next page:
To prove the statement, we partition the spanning trees of $C$ into sets $S$ and $\bar{S}$, where $S$ contains all the spanning trees that include $e$, and $\bar{S}$ is all those spanning trees that do not include $e$. We now check that every spanning tree in $S$ continues to be a spanning tree when $e$ is contracted, forming $C \circ e$, while every spanning tree in $\bar{S}$ is also a spanning tree in $C - e$. It’s also straightforward to check the converse, that every spanning tree of $C \circ e$ forms a spanning of $C$ after adding edge $e$, while every spanning tree of $C - e$ is also a spanning tree of $C$. We thus have a partition procedure for spanning trees that preserves the number of elements and the two partitions are again defined as sets of spanning trees, now on graphs having one less edge. We iterate this procedure, always applying it to edges that are part of a cycle. When we encounter a graph $C$ without cycles it is a tree, and no further iterations are necessary because for trees $\tau(C) = 1$.

We’ve described a recursive algorithm for counting the number of spanning trees. It is not very efficient because it explicitly generates one tree for every spanning tree of the original graph, a number that can grow exponentially in the number of vertices. The matrix-tree theorem gives an efficient method for calculating the same number, because the evaluation of a determinant grows only as a polynomial in the size of the matrix (number of vertices).

(b) The proof uses linearity in the expansion of determinants by minors. We will illustrate for the case of the “diamond” graph $C$ with five edges shown above. The reduced Laplacian for this graph is

$$\Delta_1(C) = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$
The rows and columns correspond to vertices 2, 3 and 4. When the edge \( e \) between 1 and 2 is removed, all that happens is that the diagonal element associated with vertex 2 is diminished by 1:

\[
\Delta_1(C - e) = \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & 0 \\
-1 & 0 & 2
\end{bmatrix}.
\]

The merging procedure is even simpler: because the resulting matrix has also row/column 2 deleted we just copy the submatrix associated with vertices 3 and 4 (since merging 1 and 2 does not affect the entries for 3 and 4):

\[
\Delta_1(C \circ e) = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}.
\]

Let’s evaluate \( \Delta_1(C) \) and \( \Delta_1(C - e) \) by expressing them as a sum over minors along their first row. They differ only in the first minor, where \( \Delta_1(C) \) has 3 as a multiplier while \( \Delta_1(C - e) \) has a 2. The difference of these determinants is thus \( 3 - 2 = 1 \) times the determinant of the \( 2 \times 2 \) minor associated with the (1,1) element, but that is exactly \( \Delta_1(C \circ e) \). In case you were wondering, the equation \( \det \Delta_1(C) = \det \Delta_1(C - e) + \det \Delta_1(C \circ e) \) evaluates to \( 8 = 4 + 4 \) in our example, and you should check that these are the correct numbers of spanning trees.

It’s easy to see that the general case calls on exactly the same determinant properties we have used in this particular example.

(c) This came up in the discussion to our solution of part (a).

3. The \( (n-1) \times (n-1) \) matrix \( \Delta_1(K_n) \) has \( (n-1) \)'s on the diagonal and \(-1\) elsewhere. One eigenvector is the all 1’s vector, with eigenvalue 1. There are \( n-2 \) eigenvectors orthogonal to this one, and all of these have eigenvalue value \( n \) (\( \Delta_1(K_n) \) equals \( n \) times the identity matrix plus the all \(-1\)'s matrix, the latter giving zero when applied to anything but multiples of the all 1’s vector). The determinant is therefore \( 1 \times n^{n-2} \).

4. This problem requires just a slight modification of the solution for the analogous problem in 2D worked out in lecture. The region in space defined by two “core” outposts that are mutual nearest neighbors, and is free of other outposts, is the union of two spheres of radius \( r \) with centers separated by \( r \). By some high school geometry we find the area of this region to be \( br^3 \), where \( b = 9\pi/4 \). Also analogously, the probability that an outpost is a core outpost is given by the integral

\[
p_{\text{core}} = \int_0^\infty 4\pi r^2 \, dr \, e^{-\rho br^3} = \frac{4\pi}{3b},
\]
where $\rho$ is the density of outposts in space. Since there are exactly two core outposts per community, the mean number of outposts per community is

$$\frac{2}{p_{\text{core}}} = \frac{27}{8} = 3.375.$$