Assignment 9 solutions

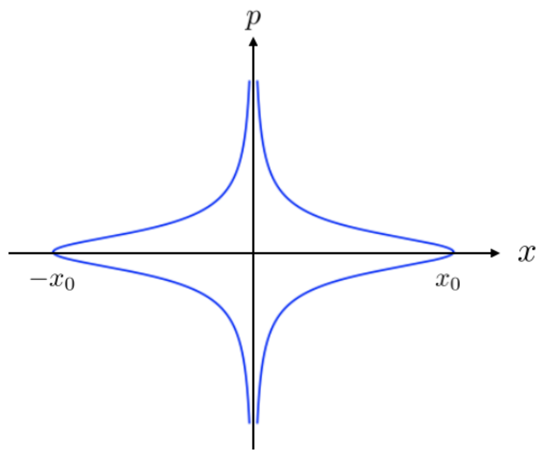
1. The Hamilton's equations of motion are given by

$$\dot{x} = \frac{\partial H}{\partial p} = p/m \qquad \dot{p} = -\frac{\partial H}{\partial x} = \begin{cases} -A/x^2, & x > 0 \\ A/x^2, & x < 0 \end{cases} .$$

When the total energy of the system is given by $E = -A/x_0$, we can express the momentum as

$$p = \pm\sqrt{2mA}\sqrt{\frac{1}{|x|} - \frac{1}{x_0}},$$

and plot the orbit in phase space. From the equation $\dot{x} = p/m$, we can see the direction x changes depends on the sign of p, which gives a clockwise flow of the orbit.



Because the orbit is symmetric to the x and y axes, the phase-space area can be written as

$$\begin{aligned} 2\pi I &= \oint p\, dq \\ &= 4\int_0^{x_0} p\, dx \\ &= 4\int_0^{x_0} \sqrt{2mA}\sqrt{\frac{1}{x} - \frac{1}{x_0}}\, dx \\ &= 2\pi\sqrt{2mAx_0} \\ &= 2\pi A\sqrt{\frac{2m}{-E}} . \end{aligned}$$

This gives us $I = A\sqrt{-2m/E}$. We can further rewrite E as

$$E = -\frac{2mA^2}{I^2} = -\frac{me^4}{8\pi^2\epsilon_0^2\, I^2}\,.$$

If I is quantized in integer multiples of Planck's constant, we obtain the energy levels

$$E(n) = -\frac{me^4}{8\pi^2\epsilon_0^2\, h^2}\,\frac{1}{n^2}\,,$$

which differ from the Rydberg series derived from the Bohr model only by a constant factor.

2. Because the function $F(q, Q)$ is a generating function of the type F_1, we have

$$p = \frac{\partial F}{\partial q} = 1/Q \qquad P = -\frac{\partial F}{\partial Q} = q/Q^2.$$

The transformed coordinates Q and P are hence given by

$$Q = 1/p \qquad P = qp^2\,,$$

and the transformed Hamiltonian $H'(P, Q)$ can be written as

$$\begin{aligned} H'(P, Q) &= H(p, q) + \frac{\partial F}{\partial t} \\ &= H(p, q) \\ &= CP. \end{aligned}$$

We can subsequently write down the Hamilton's equations of motion for Q and P as

$$\dot{Q} = \frac{\partial H'}{\partial P} = C \qquad \dot{P} = -\frac{\partial H'}{\partial Q} = 0\,,$$

and solve Q and P as

$$Q = Ct + Q_0 \qquad P = P_0\,,$$

where Q_0 and P_0 are two constants determined by the initial conditions. From these, we obtain the solutions for q and p as

$$q = P_0\,(Ct + Q_0)^2 \qquad p = 1/(Ct + Q_0)\,.$$

3. Given the Hamiltonian

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + x^2y^2 + \frac{1}{2}A(x^2 + y^2)\,,$$

we have the Hamilton's equations of motion

$$\dot{x} = p_x \qquad \dot{p_x} = -2xy^2 - Ax$$
$$\dot{y} = p_y \qquad \dot{p_y} = -2x^2y - Ay\ .$$

Plugging in the expressions of p_x and p_y, we obtain the system of second order ordinary differential equations

$$\ddot{x} = -2xy^2 - Ax$$
$$\ddot{y} = -2x^2y - Ay\,,$$

from which we could numerically calculate the trajectory of the particle using a differential equation solver.

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s = NDSolve[{x''[t] == -2 x[t] y[t]^2 - 0.1 x[t],
    y''[t] == -2 x[t]^2 y[t] - 0.1 y[t], x[0] == 1,
    y[0] == 0, x'[0] == 0, y'[0] == 2}, {x, y}, {t, 0, 5000}];
ListPlot[Table[Evaluate[{x[t], y[t]} /. s], {t, 0, 5000}],
 PlotStyle → Black, AspectRatio → 1]
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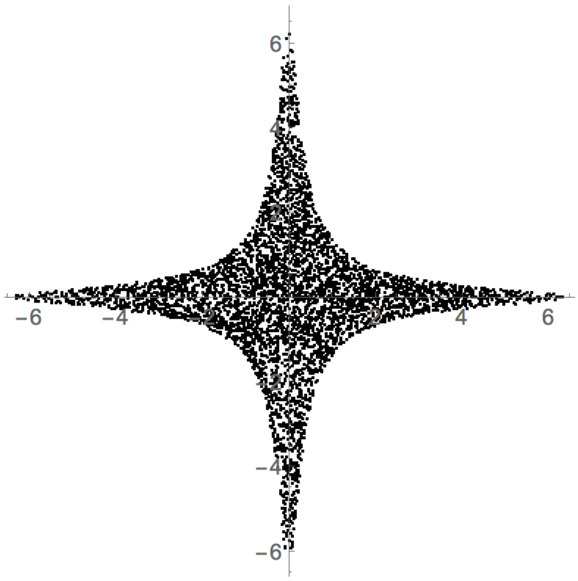

As shown in the figure above, the particle visits the accessible position space uniformly. In the next assignment, you will be asked to explicitly calculate the density of states to verify this result.