Assignment 8 solutions

1. The first order variation $\delta S$ about the extremal path $(q^*, p^*)$ can be written as

$$\delta S = \int_{t_1}^{t_2} \left( \sum_{k=1}^{N} \dot{q}_k^* \delta p_k + p_k^* \delta q_k - \frac{\partial H}{\partial q_k} \big|_{(q^*, p^*)} \delta q_k - \frac{\partial H}{\partial p_k} \big|_{(q^*, p^*)} \delta p_k \right) dt.$$ 

Integrating by part the second term, we have

$$\int_{t_1}^{t_2} p_k^* \delta \dot{q}_k \, dt = p_k^* \delta q_k \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_k^* \delta q_k \, dt$$

$$= - \int_{t_1}^{t_2} \dot{p}_k^* \delta q_k \, dt$$

because of the boundary condition that $\delta q(t_1) = \delta q(t_2) = 0$.

Therefore,

$$\delta S = \int_{t_1}^{t_2} \left( \sum_{k=1}^{N} \left[ \dot{q}_k^* - \frac{\partial H}{\partial p_k} \big|_{(q^*, p^*)} \right] \delta p_k - \left[ \dot{p}_k^* + \frac{\partial H}{\partial q_k} \big|_{(q^*, p^*)} \right] \delta q_k \right) dt,$$

which shows that $\delta S$ vanishes for independent variations of all $2N$ $\delta q$'s and $\delta p$'s if the $2N$ functions $q^*$ and $p^*$ satisfy Hamilton equations. Notice that it’s not necessary to specify restrictions on the variations $\delta p$'s, and the variation principle is still valid even when the Hamiltonian has explicit time dependence.

2. From the chain rule,

$$\dot{Q} = \frac{\partial Q}{\partial q} \dot{q} + \frac{\partial Q}{\partial p} \dot{p},$$

$$\dot{P} = \frac{\partial P}{\partial q} \dot{q} + \frac{\partial P}{\partial p} \dot{p},$$

$$\frac{\partial \dot{H}}{\partial P} = \frac{\partial H}{\partial q} \frac{\partial q}{\partial P} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial P} = -\dot{p} \frac{\partial q}{\partial P} + \dot{q} \frac{\partial p}{\partial P},$$

$$\frac{\partial \dot{H}}{\partial Q} = \frac{\partial H}{\partial q} \frac{\partial q}{\partial Q} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial Q} = -\dot{p} \frac{\partial q}{\partial Q} + \dot{q} \frac{\partial p}{\partial Q}.$$

Therefore,

$$\frac{\partial P}{\partial p} \left( \dot{Q} - \frac{\partial \dot{H}}{\partial P} \right) - \frac{\partial Q}{\partial p} \left( \dot{P} + \frac{\partial \dot{H}}{\partial Q} \right)$$

$$= \left( \frac{\partial Q}{\partial q} \frac{\partial p}{\partial q} - \frac{\partial P}{\partial q} \frac{\partial p}{\partial q} \right) \dot{q} - \left( \frac{\partial Q}{\partial p} \frac{\partial p}{\partial p} + \frac{\partial Q}{\partial p} \frac{\partial q}{\partial p} \right) \dot{q} + \left( \frac{\partial Q}{\partial P} \frac{\partial p}{\partial p} + \frac{\partial Q}{\partial Q} \frac{\partial p}{\partial p} \right) \dot{p}.$$
Notice that
\[ \frac{\partial p}{\partial P} \frac{\partial P}{\partial p} + \frac{\partial p}{\partial Q} \frac{\partial Q}{\partial p} = \frac{\partial p}{\partial p} = 1 \]
\[ \frac{\partial q}{\partial P} \frac{\partial P}{\partial q} + \frac{\partial q}{\partial Q} \frac{\partial Q}{\partial p} = \frac{\partial q}{\partial p} = 0, \]
so
\[ \frac{\partial P}{\partial p} \left( \dot{Q} - \frac{\partial \tilde{H}}{\partial P} \right) - \frac{\partial Q}{\partial p} \left( \dot{P} + \frac{\partial \tilde{H}}{\partial Q} \right) = \left[ \left( \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} \right) - 1 \right] \dot{q}. \] (1)

Similarly,
\[ \frac{\partial P}{\partial q} \left( \dot{Q} - \frac{\partial \tilde{H}}{\partial P} \right) - \frac{\partial Q}{\partial q} \left( \dot{P} + \frac{\partial \tilde{H}}{\partial Q} \right) = - \left[ \left( \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} \right) - 1 \right] \dot{p}. \] (2)

For a transformation \((Q(q,p), P(q,p))\) to preserve the form of Hamilton equations
\[ \dot{Q} = \frac{\partial \tilde{H}}{\partial P}, \quad \dot{P} = - \frac{\partial \tilde{H}}{\partial Q}, \]
its Jacobian must be unity:
\[ \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} = 1. \]

If a transformation \((Q(q,p), P(q,p))\) has unit Jacobian, Equation (1) and (2) become
\[ \frac{\partial P}{\partial p} \left( \dot{Q} - \frac{\partial \tilde{H}}{\partial P} \right) - \frac{\partial Q}{\partial p} \left( \dot{P} + \frac{\partial \tilde{H}}{\partial Q} \right) = 0 \]
\[ \frac{\partial P}{\partial q} \left( \dot{Q} - \frac{\partial \tilde{H}}{\partial P} \right) - \frac{\partial Q}{\partial q} \left( \dot{P} + \frac{\partial \tilde{H}}{\partial Q} \right) = 0. \]

These two equations can be further expressed by a matrix equation \(Ax = 0\), where
\[ A = \begin{bmatrix} \frac{\partial P}{\partial p} & -\frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & -\frac{\partial Q}{\partial q} \end{bmatrix}, \quad x = \begin{bmatrix} \dot{Q} - \frac{\partial \tilde{H}}{\partial P} \\ \dot{P} + \frac{\partial \tilde{H}}{\partial Q} \end{bmatrix}. \]

Because the determinant of \(A\) is nonzero:
\[ \det(A) = -\left( \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} \right) = -1, \]
\(x\) must vanish to satisfy the matrix equation, and this gives the form invariance of Hamiltonian equations.

Therefore, a transformation \((Q(q,p), P(q,p))\) preserves the form of Hamiltonian equations if and only if its Jacobian equals to unity. Such transformations are called “canonical transformation”. 
3. To first order of $\Delta t$,

$$Q(q(0), p(0)) = q(\Delta t) = q(0) + \Delta t \dot{q}(0)$$
$$P(q(0), p(0)) = p(\Delta t) = p(0) + \Delta t \dot{p}(0).$$

The Jacobian

$$\frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} = (1 + \Delta t \frac{\partial \dot{q}}{\partial q})(1 + \Delta t \frac{\partial \dot{p}}{\partial p}) - (\Delta t \frac{\partial \dot{p}}{\partial q})(\Delta t \frac{\partial \dot{q}}{\partial p})$$

$$= 1 + \Delta t \left( \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} \right) + O(\Delta t^2)$$
$$= 1 + \Delta t \left( \frac{\partial}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial}{\partial p} \frac{\partial H}{\partial q} \right) + O(\Delta t^2)$$
$$= 1 + O(\Delta t^2).$$

Therefore, time evolution is a canonical transformation to first order in $\Delta t$.

4. From the chain rule,

$$\dot{A} = \frac{\partial A}{\partial q} \dot{q} + \frac{\partial A}{\partial p} \dot{p} + \frac{\partial A}{\partial t}$$
$$= \frac{\partial A}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial A}{\partial t}$$

$$= \{A, H\} + \frac{\partial A}{\partial t},$$

and we have

$$\dot{q} = \{q, H\} = \frac{\partial H}{\partial p}$$
$$\dot{p} = \{p, H\} = -\frac{\partial H}{\partial q}$$
$$\dot{H} = \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.$$

For a function $I(p, q)$ to be conserved,

$$0 = \dot{I} = \{I, H\}.$$