Assignment 8 solutions

1. The first order variation δS about the extremal path $(q^\star,\ p^\star)$ can be written as

$$\delta S = \int_{t_1}^{t_2} \left(\sum_{k=1}^{N} \dot{q}_k^\star\, \delta p_k + p_k^\star\, \delta \dot{q}_k - \frac{\partial H}{\partial q_k}\bigg|_{(q^\star,\ p^\star)} \delta q_k - \frac{\partial H}{\partial p_k}\bigg|_{(q^\star,\ p^\star)} \delta p_k \right) dt.$$

Integrating by part the second term, we have

$$\begin{aligned} \int_{t_1}^{t_2} p_k^\star\, \delta \dot{q}_k\ dt &= p_k^\star\, \delta q_k \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_k^\star\, \delta q_k\ dt \\ &= -\int_{t_1}^{t_2} \dot{p}_k^\star\, \delta q_k\ dt \end{aligned}$$

because of the boundary condition that $\delta q(t_1) = \delta q(t_2) = 0$.

Therefore,

$$\delta S = \int_{t_1}^{t_2} \left(\sum_{k=1}^{N} \left[\dot{q}_k^\star - \frac{\partial H}{\partial p_k}\bigg|_{(q^\star,\ p^\star)} \right] \delta p_k - \left[\dot{p}_k^\star + \frac{\partial H}{\partial q_k}\bigg|_{(q^\star,\ p^\star)} \right] \delta q_k \right) dt,$$

which shows that δS vanishes for independent variations of all $2N$ δq's and δp's if the $2N$ functions $q^\star$ and $p^\star$ satisfy Hamilton equations. Notice that it's not necessary to specify restrictions on the variations δp's, and the variation principle is still valid even when the Hamiltonian has explicit time dependence.

2. From the chain rule,

$$\begin{aligned} \dot{Q} &= \frac{\partial Q}{\partial q}\, \dot{q} + \frac{\partial Q}{\partial p}\, \dot{p} \\ \dot{P} &= \frac{\partial P}{\partial q}\, \dot{q} + \frac{\partial P}{\partial p}\, \dot{p} \\ \frac{\partial \tilde{H}}{\partial P} &= \frac{\partial H}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial P} = -\dot{p}\, \frac{\partial q}{\partial P} + \dot{q}\, \frac{\partial p}{\partial P} \\ \frac{\partial \tilde{H}}{\partial Q} &= \frac{\partial H}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial Q} = -\dot{p}\, \frac{\partial q}{\partial Q} + \dot{q}\, \frac{\partial p}{\partial Q}\ . \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{\partial P}{\partial p}\left(\dot{Q} - \frac{\partial \tilde{H}}{\partial P} \right) - \frac{\partial Q}{\partial p}\left(\dot{P} + \frac{\partial \tilde{H}}{\partial Q} \right) \\ &= \left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} \right) \dot{q} - \left(\frac{\partial p}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial p}{\partial Q}\frac{\partial Q}{\partial p} \right) \dot{q} + \left(\frac{\partial q}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial q}{\partial Q}\frac{\partial Q}{\partial p} \right) \dot{p}\ . \end{aligned}$$

Notice that

$$\frac{\partial p}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial p}{\partial Q}\frac{\partial Q}{\partial p} = \frac{\partial p}{\partial p} = 1$$
$$\frac{\partial q}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial q}{\partial Q}\frac{\partial Q}{\partial p} = \frac{\partial q}{\partial p} = 0\,,$$

so

$$\frac{\partial P}{\partial p}\left(\dot{Q} - \frac{\partial \tilde{H}}{\partial P}\right) - \frac{\partial Q}{\partial p}\left(\dot{P} + \frac{\partial \tilde{H}}{\partial Q}\right) = \left[\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\right) - 1\right]\,\dot{q}\,. \tag{1}$$

Similarly,

$$\frac{\partial P}{\partial q}\left(\dot{Q} - \frac{\partial \tilde{H}}{\partial P}\right) - \frac{\partial Q}{\partial q}\left(\dot{P} + \frac{\partial \tilde{H}}{\partial Q}\right) = -\left[\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\right) - 1\right]\,\dot{p}\,. \tag{2}$$

For a transformation $(Q(q,p),\ P(q,p))$ to preserve the form of Hamilton equations

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} \qquad \dot{P} = -\frac{\partial \tilde{H}}{\partial Q}\,,$$

its Jacobian must be unity:

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1.$$

If a transformation $(Q(q,p),\ P(q,p))$ has unit Jacobian, Equation (1) and (2) become

$$\frac{\partial P}{\partial p}\left(\dot{Q} - \frac{\partial \tilde{H}}{\partial P}\right) - \frac{\partial Q}{\partial p}\left(\dot{P} + \frac{\partial \tilde{H}}{\partial Q}\right) = 0$$
$$\frac{\partial P}{\partial q}\left(\dot{Q} - \frac{\partial \tilde{H}}{\partial P}\right) - \frac{\partial Q}{\partial q}\left(\dot{P} + \frac{\partial \tilde{H}}{\partial Q}\right) = 0\,.$$

These two equations can be further expressed by a matrix equation $Ax = 0$, where

$$A = \begin{bmatrix} \frac{\partial P}{\partial p} & -\frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & -\frac{\partial Q}{\partial q} \end{bmatrix} \qquad x = \begin{bmatrix} \dot{Q} - \frac{\partial \tilde{H}}{\partial P} \\ \dot{P} + \frac{\partial \tilde{H}}{\partial Q} \end{bmatrix}\,.$$

Because the determinant of A is nonzero:

$$\det(A) = -(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}) = -1,$$

x must vanish to satisfy the matrix equation, and this gives the form invariance of Hamiltonian equations.

Therefore, a transformation $(Q(q,p),\ P(q,p))$ preserves the form of Hamiltonian equations if and only if its Jacobian equals to unity. Such transformations are called "canonical transformation".

3. To first order of Δt,

$$Q(q(0),\ p(0)) = q(\Delta t) = q(0) + \Delta t\ \dot{q}(0)$$
$$P(q(0),\ p(0)) = p(\Delta t) = p(0) + \Delta t\ \dot{p}(0)\ .$$

The Jacobian

$$\begin{aligned}\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} &= (1 + \Delta t\ \frac{\partial \dot{q}}{\partial q})(1 + \Delta t\ \frac{\partial \dot{p}}{\partial p}) - (\Delta t\ \frac{\partial \dot{p}}{\partial q})(\Delta t\ \frac{\partial \dot{q}}{\partial p}) \\ &= 1 + \Delta t\ (\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p}) + O(\Delta t^2) \\ &= 1 + \Delta t\ (\frac{\partial}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial}{\partial p}\frac{\partial H}{\partial q}) + O(\Delta t^2) \\ &= 1 + O(\Delta t^2)\ .\end{aligned}$$

Therefore, time evolution is a canonical transformation to first order in Δt.

4. From the chain rule,

$$\begin{aligned}\dot{A} &= \frac{\partial A}{\partial q}\ \dot{q} + \frac{\partial A}{\partial p}\ \dot{p} + \frac{\partial A}{\partial t} \\ &= \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial A}{\partial t} \\ &= \{A, H\} + \frac{\partial A}{\partial t}\ ,\end{aligned}$$

and we have

$$\begin{aligned}\dot{q} &= \{q, H\} = \frac{\partial H}{\partial p} \\ \dot{p} &= \{p, H\} = -\frac{\partial H}{\partial q} \\ \dot{H} &= \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}\ .\end{aligned}$$

For a function $I(p, q)$ to be conserved,

$$0 = \dot{I} = \{I, H\}\ .$$