Assignment 7 solutions

1. Under the continuous transformation with the parameter $s$, the Lagrangian can be expressed as

$$L(s) = L(q_1(t+s), \cdots, q_N(t+s); \dot{q}_1(t+s), \cdots, \dot{q}_N(t+s)).$$

Since $L$ is constructed from functions ($q$’s and $\dot{q}$’s) all having the argument $s + t$, by the chain rule of calculus we can infer that

$$\frac{dL}{ds}
|_{s=0} = \sum_{k=1}^{N} \left( \frac{\partial L}{\partial q_k} \dot{q}_k + \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k \right) = \frac{dL}{dt}. \quad (1)$$

We can hence identify the function $F$ as the Lagrangian $L$, apart from an irrelevant constant.

Using the Euler-Lagrange equation, we can rewrite (1) as

$$\sum_{k=1}^{N} \left( \frac{\partial L}{\partial q_k} \dot{q}_k + \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k \right) = \sum_{k=1}^{N} \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) \dot{q}_k + \frac{\partial L}{\partial q_k} \dot{q}_k \right)$$

$$= \frac{d}{dt} \left( \sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k \right)$$

$$= \frac{d}{dt} \left( \sum_{k=1}^{N} p_k \dot{q}_k \right) = \frac{dL}{dt}.$$

We can hence define a conserved quantity $I$

$$I \equiv \sum_{k=1}^{N} p_k \dot{q}_k - L$$

so that $dI/dt = 0$. This quantity $I$ is exactly the Hamiltonian $H$ we defined several weeks ago.
From Assignment 4, the Lagrangian of the system is given by

\[ L = \frac{1}{2} m (R^2 + r^2 - 2Rr \cos \theta) \dot{\theta}^2 - mg(R - r \cos \theta), \]

with the conjugate momentum

\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m(R^2 + r^2 - 2Rr \cos \theta) \dot{\theta}. \]

The Hamiltonian is hence given by

\[
H = \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L = \frac{1}{2} m (R^2 + r^2 - 2Rr \cos \theta) \dot{\theta}^2 + mg(R - r \cos \theta)
\]

\[ = \frac{p_\theta^2}{2m(R^2 + r^2 - 2Rr \cos \theta)} + mg(R - r \cos \theta), \]

Because \( \frac{dH}{dt} = \frac{\partial L}{\partial t} = 0 \), we can plot the trajectory that stays on the surface of constant energy \( E \)

\[ H(\theta, p_\theta) = \frac{p_\theta^2}{2m(R^2 + r^2 - 2Rr \cos \theta)} + mg(R - r \cos \theta) = E. \]
When \( E = mg(R - r) \), the “surface” of constant energy stays at the origin, which corresponds to the trivial state that the point mass stays at the lowest point forever. For \( mg(R - r) < E < mg(R + r) \), the trajectory is a bounded oscillation, with turning points at \( \theta_0 = \pm \cos^{-1}((mgR - E)/(mgr)) \). For \( E > mg(R + r) \), the total energy is large enough for the wheel to keep moving forward/backward. Since this motion has a period \( 2\pi \), the plot with the range \((-\pi, \pi)\) of \( \theta \) is enough for expressing the motion.

The direction of the flow in phase space can be determined by the equation

\[
\dot{\theta} = \frac{p_\theta}{m(R^2 + r^2 - 2Rr \cos \theta)},
\]

which indicates that the change of \( \theta \) depends on the sign of \( p_\theta \).

From the discussion above, we know that the change of the topology of the orbits occurs at \( E^* = mg(R + r) \).

3. Switching to the dimensionless time, the equations of motion become

\[
\begin{align*}
\cos \theta \ddot{\theta} &= \sin \theta \dot{\alpha}^2 - \cos^2 \theta \cos \alpha \\
\cos \theta \dot{\alpha} &= -\left(\frac{\sin \theta + 2 \cos^2 \theta}{\sin \theta}\right) \dot{\theta} \dot{\alpha} + \frac{\cos^3 \theta}{\sin \theta} \sin \alpha.
\end{align*}
\]

In the small angle approximation where \( \theta(t) = \pi/2 + \beta(t) \) and both \( \beta(t) \) and \( \alpha(t) \) are small, we can expand \( \cos \theta \) and \( \sin \theta \) as

\[
\begin{align*}
\cos \theta &= \cos(\pi/2 + \beta) = -\sin \beta = -\beta + O(\beta^3) \\
\sin \theta &= \sin(\pi/2 + \beta) = \cos \beta = 1 - \beta^2/2 + O(\beta^4).
\end{align*}
\]

Substituting these expressions back to the equations of motion and only keeping the relevant lowest-order terms, we have

\[
\begin{align*}
-\beta \ddot{\beta} &= \dot{\alpha}^2 - \beta^2 \\
\beta \ddot{\alpha} &= \ddot{\beta} \dot{\alpha}.
\end{align*}
\]

We first observe that the system of equations becomes

\[
\begin{align*}
\beta \ddot{\beta} &= \beta^2 \\
0 &= 0
\end{align*}
\]

when \( \dot{\alpha} = 0 \), which gives us the first two types of solutions:

(a) \( \alpha = \alpha_0 \) and \( \beta = 0 \), where \( \alpha_0 \) is a small constant. The resulting potential energy \( V = \epsilon \sin \theta \cos \alpha \) remains constant. From energy conservation we can see that this solution amounts to the parallel transport of the dipole around the equator of the sphere at constant angular velocity \( \phi \).
(b) $\alpha = \alpha_0$ and $\ddot{\beta} = \beta$. We can solve $\beta(t) = \beta_0 e^{-t}$. Substituting $\alpha(t)$ and $\beta(t)$ back to the non-holonomic constraint

$$\dot{\alpha} + \cos \theta \dot{\phi} = 0,$$

we have $\dot{\phi} = 0$ and therefore $\phi(t) = \phi_0$. This solution describes the longitudinal motion of the dipole towards the equator of the sphere.

For $\dot{\alpha} \neq 0$, $\beta \neq 0$, and we can rearrange Equation (3) to

$$\frac{\ddot{\beta}}{\dot{\beta}} = \frac{\dot{\alpha}}{\dot{\alpha}},$$

which gives the relation

$$\dot{\alpha}(t) = c \beta(t).$$

Substituting this relation to the non-holonomic constraint, we have

$$\beta(t)(c - \dot{\phi}) = 0,$$

and we can solve $\phi(t) = ct + \phi_0$. Also, Equation (2) becomes

$$\ddot{\beta} = -(c^2 - 1) \beta,$$

from which we obtain the other three types of solutions:

(c) $c > 1$:

$$\beta(t) = \beta_1 \cos(\sqrt{c^2 - 1} t) + \beta_2 \sin(\sqrt{c^2 - 1} t).$$

In this solution, the dipole moves longitudinally at constant angular velocity $\dot{\phi} = c$ while oscillates sinusoidally about the equator in the latitudinal direction.

(d) $c = 1$:

$$\beta(t) = \beta_1 t + \beta_2.$$

This solution describes a dipole moving longitudinally at constant angular velocity $\dot{\phi} = c$ while moving away from the equator at constant rate $\beta_1$. We notice that this solution fails to describe the motion of the dipole once $\beta(t)$ violates the small angle approximation.

(e) $c < 1$:

$$\beta(t) = \beta_1 \exp(\sqrt{1 - c^2} t) + \beta_2 \exp(-\sqrt{1 - c^2} t).$$

The dipole also moves longitudinally at constant angular velocity $\dot{\phi} = c$ whereas moves away from the equator at an exponential rate. This solution also becomes invalid once $\beta(t)$ breaks the small angle approximation.

Notice that when $c \rightarrow 0$,

$$\dot{\alpha}(t) \rightarrow 0$$

$$\beta(t) \rightarrow \beta_1 e^t + \beta_2 e^{-t},$$

which gives us the solutions obtained in (a) ($\beta_1 = \beta_2 = 0$) and (b) ($\beta_1 = 0$).