Assignment 7 solutions

1. Under the continuous transformation with the parameter s, the Lagrangian can be expressed as

$$L(s) = L(q_1(t+s), \cdots, q_N(t+s); \dot{q}_1(t+s), \cdots, \dot{q}_N(t+s)).$$

Since L is constructed from functions (q's and $\dot{q}$'s) all having the argument $s+t$, by the chain rule of calculus we can infer that

$$\left.\frac{dL}{ds}\right|_{s=0} = \sum_{k=1}^{N} \left.\left(\frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k \right)\right|_t = \frac{dL}{dt}. \tag{1}$$

We can hence identify the function F as the Lagrangian L, apart from an irrelevant constant.

Using the Euler-Lagrange equation, we can rewrite (1) as

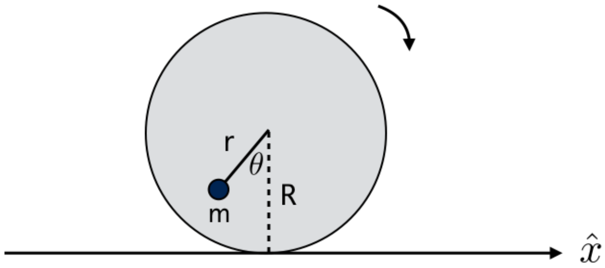
$$\begin{aligned}
\sum_{k=1}^{N} \left(\frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k \right) &= \sum_{k=1}^{N} \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k \right) \\
&= \frac{d}{dt}\left(\sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k \right) \\
&= \frac{d}{dt}\left(\sum_{k=1}^{N} p_k \dot{q}_k \right) = \frac{dL}{dt}.
\end{aligned}$$

We can hence define a conserved quantity I

$$I \equiv \sum_{k=1}^{N} p_k \dot{q}_k - L$$

so that $dI/dt = 0$. This quantity I is exactly the Hamiltonian H we defined several weeks ago.

2.



From Assignment 4, the Lagrangian of the system is given by

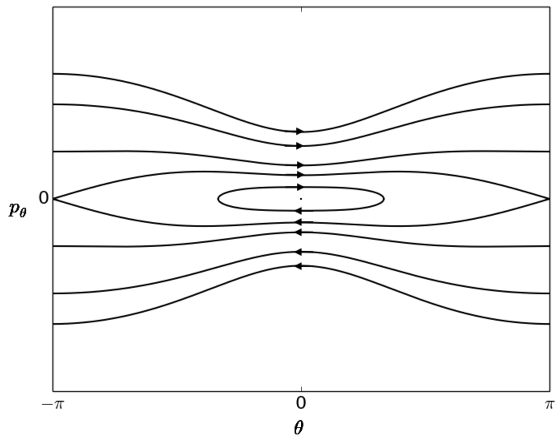
$$L = \frac{1}{2}m(R^2 + r^2 - 2Rr\cos\theta)\,\dot{\theta}^2 - mg(R - r\cos\theta)\,,$$

with the conjugate momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m(R^2 + r^2 - 2Rr\cos\theta)\,\dot{\theta}\ .$$

The Hamiltonian is hence given by

$$\begin{aligned} H &= \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L \\ &= \frac{1}{2}m(R^2 + r^2 - 2Rr\cos\theta)\,\dot{\theta}^2 + mg(R - r\cos\theta) \\ &= \frac{p_\theta^2}{2m(R^2 + r^2 - 2Rr\cos\theta)} + mg(R - r\cos\theta) \end{aligned}$$



Because $dH/dt = \partial L/\partial t = 0$, we can plot the trajectory that stays on the surface of constant energy E

$$H(\theta, p_\theta) = \frac{p_\theta^2}{2m(R^2 + r^2 - 2Rr\cos\theta)} + mg(R - r\cos\theta) = E.$$

When $E = mg(R - r)$, the "surface" of constant energy stays at the origin, which corresponds to the trivial state that the point mass stays at the lowest point forever. For $mg(R - r) < E < mg(R + r)$, the trajectory is a bounded oscillation, with turning points at $\theta_0 = \pm\cos^{-1}((mgR - E)/(mgr))$. For $E > mg(R + r)$, the total energy is large enough for the wheel to keep moving forward/backward. Since this motion has a period 2π, the plot with the range $(-\pi, \pi)$ of θ is enough for expressing the motion.

The direction of the flow in phase space can be determined by the equation

$$\dot{\theta} = \frac{p_\theta}{m(R^2 + r^2 - 2Rr\cos\theta)} ,$$

which indicates that the change of θ depends on the sign of p_θ.

From the discussion above, we know that the change of the topology of the orbits occurs at $E^* = mg(R + r)$.

3. Switching to the dimensionless time, the equations of motion become

$$\begin{aligned}\cos\theta\,\ddot{\theta} &= \sin\theta\,\dot{\alpha}^2 - \cos^2\theta\cos\alpha\\ \cos\theta\,\ddot{\alpha} &= -\left(\sin\theta + 2\,\frac{\cos^2\theta}{\sin\theta}\right)\dot{\theta}\dot{\alpha} + \frac{\cos^3\theta}{\sin\theta}\sin\alpha\ .\end{aligned}$$

In the small angle approximation where $\theta(t) = \pi/2 + \beta(t)$ and both $\beta(t)$ and $\alpha(t)$ are small, we can expand $\cos\theta$ and $\sin\theta$ as

$$\begin{aligned}\cos\theta &= \cos(\pi/2 + \beta) = -\sin\beta = -\beta + O(\beta^3)\\ \sin\theta &= \sin(\pi/2 + \beta) = \cos\beta = 1 - \beta^2/2 + O(\beta^4).\end{aligned}$$

Substituting these expressions back to the equations of motion and only keeping the relevant lowest-order terms, we have

$$-\beta\ddot{\beta} = \dot{\alpha}^2 - \beta^2 \tag{2}$$
$$\beta\ddot{\alpha} = \dot{\beta}\dot{\alpha}. \tag{3}$$

We first observe that the system of equations becomes

$$\begin{aligned}\beta\ddot{\beta} &= \beta^2\\ 0 &= 0\end{aligned}$$

when $\dot{\alpha} = 0$, which gives us the first two types of solutions:

(a) $\alpha = \alpha_0$ and $\beta = 0$, where α_0 is a small constant. The resulting potential energy $V = \epsilon\sin\theta\cos\alpha$ remains constant. From energy conservation we can see that this solution amounts to the parallel transport of the dipole around the equator of the sphere at constant angular velocity $\dot{\phi}$.

(b) $\alpha = \alpha_0$ and $\ddot{\beta} = \beta$. We can solve $\beta(t) = \beta_0 e^{-t}$. Substituting $\alpha(t)$ and $\beta(t)$ back to the non-holonomic constraint

$$\dot{\alpha} + \cos\theta\, \dot{\phi} = 0\,,$$

we have $\dot{\phi} = 0$ and therefore $\phi(t) = \phi_0$. This solution describes the longitudinal motion of the dipole towards the equator of the sphere.

For $\dot{\alpha} \neq 0$, $\beta \neq 0$, and we can rearrange Equation (3) to

$$\frac{\dot{\beta}}{\beta} = \frac{\ddot{\alpha}}{\dot{\alpha}},$$

which gives the relation

$$\dot{\alpha}(t) = c\beta(t).$$

Substituting this relation to the non-holonomic constraint, we have

$$\beta(t)(c - \dot{\phi}) = 0,$$

and we can solve $\phi(t) = ct + \phi_0$. Also, Equation (2) becomes

$$\ddot{\beta} = -(c^2 - 1)\beta,$$

from which we obtain the other three types of solutions:

(c) $c > 1$:

$$\beta(t) = \beta_1 \cos(\sqrt{c^2 - 1}\; t) + \beta_2 \sin(\sqrt{c^2 - 1}\; t).$$

In this solution, the dipole moves longitudinally at constant angular velocity $\dot{\phi} = c$ while oscillates sinusoidally about the equator in the latitudinal direction.

(d) $c = 1$:

$$\beta(t) = \beta_1 t + \beta_2.$$

This solution describes a dipole moving longitudinally at constant angular velocity $\dot{\phi} = c$ while moving away from the equator at constant rate β_1. We notice that this solution fails to describe the motion of the dipole once $\beta(t)$ violates the small angle approximation.

(e) $c < 1$:

$$\beta(t) = \beta_1 \exp(\sqrt{1 - c^2}\; t) + \beta_2 \exp(-\sqrt{1 - c^2}\; t).$$

The dipole also moves longitudinally at constant angular velocity $\dot{\phi} = c$ whereas moves away from the equator at an exponential rate. This solution also becomes invalid once $\beta(t)$ breaks the small angle approximation.

Notice that when $c \to 0$,

$$\begin{aligned} &\dot{\alpha}(t) \to 0 \\ &\beta(t) \to \beta_1 e^t + \beta_2 e^{-t}\,, \end{aligned}$$

which gives us the solutions obtained in (a) ($\beta_1 = \beta_2 = 0$) and (b) ($\beta_1 = 0$).