Assignment 6 solutions

1. Using Equation (15.5) in the lecture, we can define the new Lagrangian that includes a sum over the constraint functions.

$$L' = \frac{1}{2}m(|\dot{\mathbf{r}}_1|^2 + |\dot{\mathbf{r}}_2|^2 + |\dot{\mathbf{r}}_3|^2) + (\lambda_1(t)|\mathbf{r}_1 - \mathbf{r}_2|^2 + \lambda_2(t)|\mathbf{r}_2 - \mathbf{r}_3|^2).$$

Substituting this into the Euler-Lagrange equation, we have the equations of motion

$$\begin{aligned}
0 &= -\frac{d}{dt}(m\dot{\mathbf{r}}_1) + 2\lambda_1(t)(\mathbf{r}_1 - \mathbf{r}_2)\\
0 &= -\frac{d}{dt}(m\dot{\mathbf{r}}_2) + 2\lambda_1(t)(\mathbf{r}_2 - \mathbf{r}_1) + 2\lambda_2(t)(\mathbf{r}_2 - \mathbf{r}_3)\\
0 &= -\frac{d}{dt}(m\dot{\mathbf{r}}_3) + 2\lambda_2(t)(\mathbf{r}_3 - \mathbf{r}_2).
\end{aligned}$$

We now follow the same procedure in the lecture to solve for the Lagrange multipliers. Taking time derivative of the constraint equations, we have

$$\begin{aligned}
0 &= \frac{d^2}{dt^2}|\mathbf{r}_1 - \mathbf{r}_2|^2 = 2(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2)\cdot(\mathbf{r}_1 - \mathbf{r}_2) + 2|\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2|^2\\
0 &= \frac{d^2}{dt^2}|\mathbf{r}_2 - \mathbf{r}_3|^2 = 2(\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_3)\cdot(\mathbf{r}_2 - \mathbf{r}_3) + 2|\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_3|^2.
\end{aligned}$$

Define $\lambda_1' \equiv 2\lambda_1/m$ and $\lambda_2' \equiv 2\lambda_2/m$, and substitute the equations of motions into the above equations.

$$\begin{aligned}
2\lambda_1' d^2 - \lambda_2'(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_2 - \mathbf{r}_3) + |\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2|^2 &= 0\\
-\lambda_1'(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_2 - \mathbf{r}_3) + 2\lambda_2' d^2 + |\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_3|^2 &= 0.
\end{aligned}$$

We can hence solve λ_1' and λ_2' as

$$\begin{aligned}
\lambda_1' &= \frac{2d^2|\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2|^2 + |\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_3|^2(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_2 - \mathbf{r}_3)}{|(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_2 - \mathbf{r}_3)|^2 - 4d^4}\\
\lambda_2' &= \frac{2d^2|\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_3|^2 + |\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2|^2(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_2 - \mathbf{r}_3)}{|(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_2 - \mathbf{r}_3)|^2 - 4d^4}.
\end{aligned}$$

Therefore, the equations of motion become

$$\begin{aligned}
\ddot{r}_1 &= \frac{2d^2|\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2|^2 + |\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_3|^2(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_2 - \mathbf{r}_3)}{|(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_2 - \mathbf{r}_3)|^2 - 4d^4}(\mathbf{r}_1 - \mathbf{r}_2)\\
\ddot{r}_3 &= \frac{2d^2|\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_3|^2 + |\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2|^2(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_2 - \mathbf{r}_3)}{|(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_2 - \mathbf{r}_3)|^2 - 4d^4}(\mathbf{r}_3 - \mathbf{r}_2)\\
\ddot{r}_2 &= -\ddot{r}_1 - \ddot{r}_3.
\end{aligned}$$

Unlike the $N = 2$ case, the Lagrange multipliers λ_1' and λ_2' are generally time-dependent.

2. (a) Integrating Equation (1) with time over one period of the orbit, we have

$$\Delta\alpha = \int \dot{\alpha}\, dt = -\int \cos\theta_0 \dot{\phi}\, dt = -\cos\theta_0\, \Delta\phi = -2\pi\cos\theta_0.$$

From Archimedes' hat-box theorem, we can calculate the surface area of the spherical cap as $2\pi r^2(1-\cos\theta_0)$, and we hence arrive at the relation between $\Delta\alpha$ and the solid angle Ω spanned by the spherical cap as

$$\Omega = 2\pi + \Delta\alpha.$$

(b) The Lagrangian of the system is given by

$$L = T - V = \frac{1}{2} I(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) - \epsilon \sin\theta\cos\alpha.$$

Our task is to extremize the action $S = \int L\, dt$ under the non-holonomic constraint (Equation (1)), which can be expressed as finite differences over time Δt by

$$\delta\alpha(t) + \cos\theta\, \delta\phi(t) = 0.$$

The equations of motion are hence given by

$$\begin{aligned}
\frac{\delta S}{\delta\theta} &= \frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = -I\ddot{\theta} + I\sin\theta\cos\theta\dot{\phi}^2 - \epsilon\cos\theta\cos\alpha = 0\\
\frac{\delta S}{\delta\phi} &= \left(\frac{\partial L}{\partial\phi} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}}\right) + \lambda(t)\cos\theta = -I\sin^2\theta\, \ddot{\phi} - 2I\sin\theta\cos\theta\, \dot{\theta}\dot{\phi} + \lambda(t)\cos\theta = 0\\
\frac{\delta S}{\delta\alpha} &= \left(\frac{\partial L}{\partial\alpha} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\alpha}}\right) + \lambda(t) = \epsilon\sin\theta\sin\alpha + \lambda(t) = 0.
\end{aligned}$$

We can immediately solve the Lagrange multiplier $\lambda(t) = -\epsilon\sin\theta\sin\alpha$. From the non-holonomic constraint, we have

$$\ddot{\alpha} = -\ddot{\phi}\cos\theta + \dot{\theta}\dot{\phi}\sin\theta = -\ddot{\phi}\cos\theta - \dot{\theta}\dot{\alpha}\tan\theta,$$

and $\ddot{\phi}$ can be expressed as

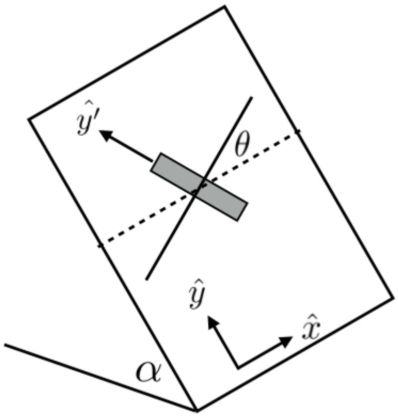
$$\ddot{\phi} = -\frac{1}{\cos\theta}(\ddot{\alpha} + \dot{\theta}\dot{\alpha}\tan\theta).$$

Substituting $\ddot{\alpha}$ and $\lambda(t)$ back to the first two equations of motion, we arrive at

$$\begin{aligned}
\cos\theta\, \ddot{\theta} &= \sin\theta\, \dot{\alpha}^2 - \frac{\epsilon}{I}\cos^2\theta\cos\alpha\\
\cos\theta\, \ddot{\alpha} &= -(\sin\theta + 2\,\frac{\cos^2\theta}{\sin\theta})\dot{\theta}\dot{\alpha} + \frac{\epsilon}{I}\frac{\cos^3\theta}{\sin\theta}\sin\alpha\,,
\end{aligned}$$

from which we can solve the time evolutions $\alpha(t)$ and $\theta(t)$ given the initial conditions.

3.



(a) Consider the axis through the wheel that is normal to the plane. Since the length of lever arms of all forces (gravity, normal force and static friction) about the axis equals to zero, the net torque goes to zero. Therefore, the angular momentum of the wheel about the axis is conserved, and the angular velocity $\dot{\theta}$ equals to a constant ω. The time dependence of $\theta(t)$ is subsequently given by

$$\theta(t) = \omega t + \theta_0,$$

where θ_0 is a constant determined by the initial condition.

(b) Define the direction perpendicular to the plane as $\hat{z}$. The gravitational force can be decomposed as

$$\vec{F}_g = -Mg \sin\alpha \hat{y} - Mg \cos\alpha \hat{z},$$

whose z component cancels with the normal force.
Defining the component of static friction along $\hat{y}'$ as F_s, the $\hat{y}'$ component of the net force is hence given by

$$F' = F_s - Mg \sin\alpha \cos\theta.$$

(c) The $\hat{y}'$ component of the net force is given by

$$\begin{aligned} F' &= M\ddot{y}' \\ &= M(\cos\theta \ddot{y} - \sin\theta \ddot{x}) \\ &= -Mr[\cos\theta(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) + \sin\theta(\ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta)] \\ &= -Mr\ddot{\phi}. \end{aligned}$$

(d) Because the length of the lever arms of gravity and normal force about the wheel axis equals to zero, these forces don't contribute to the torque. Therefore, the only contribution to the torque comes from the static friction and has a magnitude rF_s. We thus have the relation

$$I\ddot{\phi} = rF_s.$$

(e) Using the above results, the $\hat{y}'$ component of the net force can be rewritten as

$$\begin{aligned} F' &= -Mr\ddot{\phi} \\ &= F_s - Mg\sin\alpha\cos\theta \\ &= I\ddot{\phi}/r - Mg\sin\alpha\cos(\omega t + \theta_0), \end{aligned}$$

and we arrive at the relation

$$(I + Mr^2)\ddot{\phi} = Mgr\sin\alpha\cos(\omega t + \theta_0),$$

from which we can solve for the trajectories $x(t)$ and $y(t)$.