Assignment 5 solutions

1. The Hamiltonian H is given by

$$\begin{aligned}H &= \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \\ &= \sum_k \dot{q}_k \frac{\partial (T-V)}{\partial \dot{q}_k} - (T-V) \\ &= \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} - (T-V)\,,\end{aligned}$$

because V is independent of the generalized velocities. Taking the partial derivative of kinetic energy T with respect to the general velocities, we have

$$\begin{aligned}\frac{\partial T}{\partial \dot{q}_k} &= \frac{1}{2}\sum_i m_i \frac{\partial}{\partial \dot{q}_k}(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) \\ &= \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}\,.\end{aligned}$$

Since all the particle positions in the system only depend on the general coordinates,

$$\dot{\mathbf{r}}_i = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k\,,$$

and hence

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} = \frac{\partial \mathbf{r}_i}{\partial q_k}\,.$$

Substituting these back to the expression of the Hamiltonian, we obtain

$$\begin{aligned}H &= \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} - (T-V) \\ &= \sum_k \dot{q}_k \left[\sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}\right] - (T-V) \\ &= \sum_k \dot{q}_k \left[\sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\right] - (T-V) \\ &= \sum_i m_i \dot{\mathbf{r}}_i \cdot \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k - (T-V) \\ &= \sum_i m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i - (T-V) \\ &= 2T - (T-V) = T + V = E\,.\end{aligned}$$

2. We first define the 'Lagrangian' of the brachistochrone problem as

$$L(y, y'; x) = \sqrt{\frac{1 + y'^2}{-2gy}}\,,$$

which is the integrand of the functional $T[y]$. Because the Lagrangian has no explicit x dependence, its associated Hamiltonian must be a constant with respect to x. We denote this constant as $-\epsilon$. Therefore,

$$\begin{aligned} H &= y'\frac{\partial L}{\partial y'} - L \\ &= -\frac{1}{\sqrt{-2gy}\sqrt{1+y'^2}} = -\epsilon\,, \end{aligned}$$

and we obtain the first-order equation of motion

$$2gy(1 + y'^2)\epsilon^2 + 1 = 0\,.$$

With the parametrization of x and y

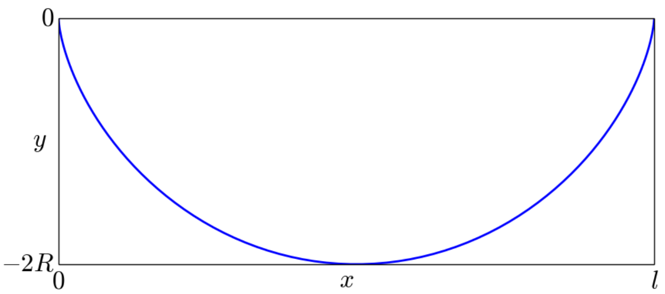
$$\begin{aligned} x &= R\theta - R\sin\theta \\ y &= -R + R\cos\theta\,, \end{aligned}$$

we have

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta}\left(\frac{dx}{d\theta}\right)^{-1} = \frac{-\sin\theta}{1-\cos\theta}\,.$$

Substituting y and y' back to the equation of motion, we can solve ϵ as

$$\epsilon = \frac{1}{2\sqrt{gR}}\,.$$



The above figure shows the brachistochrone curve with θ ranging from 0 to 2π. To have the span of x be l, we need $x(2\pi) - x(0) = 2\pi R = l$, or $R = l/(2\pi)$.

Finally, plugging the parametrization of x and y back to $L(y, y'; x)$, we have

$$L(y, y'; x) = \frac{1}{\sqrt{gR}}\frac{1}{1-\cos\theta}\,.$$

The transit-time functional thus becomes

$$\begin{aligned} T[y] &= \int_0^l L(y, y'; x) dx \\ &= \int_0^{2\pi} \left(\frac{1}{\sqrt{gR}} \frac{1}{1 - \cos\theta} \right) (R - R\cos\theta)\, d\theta \\ &= 2\pi \sqrt{\frac{R}{g}} = \sqrt{\frac{2\pi l}{g}}\,. \end{aligned}$$

3. Imagine the wall to be a mirror. Finding a minimum-time path from the point reaching the wall at a distance l away is hence equivalent to finding a minimum-time path from the point reaching another point of the same height at a distance $2l$ away, with the condition that the new path is symmetric at $x = l$. We know that the brachistochrone curve is the only path that satisfies these two conditions. Therefore, the path from the point to the wall should be the first half of the brachistochrone curve; otherwise there would be another shorter-time solution to the original problem.

   Using the result of Problem 2, the minimum time for the particle to reach the wall is therefore

$$\frac{1}{2}\sqrt{\frac{2\pi(2l)}{g}} = \sqrt{\frac{\pi l}{g}}\,.$$

   Note that we need to plug in $2l$ instead of l for the distance between the two points into the result of Problem 2.

4. (a) Substituting the Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2$ into the Euler-Lagrange equation, we can solve the equation of motion of the extremal trajectory $x^\star$ as

$$\ddot{x}^\star + \omega_0^2 x^\star = 0.$$

   The action

$$\begin{aligned} S[x] &= S[x^\star + \delta x] \\ &= \int_0^T \left[\frac{1}{2} m(\dot{x}^\star + \delta\dot{x})^2 - \frac{1}{2} m\omega_0^2 (x^\star + \delta x)^2 \right] dt \\ &= \int_0^T (\frac{1}{2} m\dot{x}^{\star 2} - \frac{1}{2} m\omega_0^2 x^{\star 2})\, dt + \int_0^T (\frac{1}{2} m\delta\dot{x}^2 - \frac{1}{2} m\omega_0^2 \delta x^2)\, dt \\ &+ \int_0^T m\dot{x}^\star \delta\dot{x}\, dt - \int_0^T m\omega_0^2 x^\star \delta x\, dt. \end{aligned}$$

   Integrating by part the third term, we have

$$\int_0^T m\dot{x}^\star \delta\dot{x}\, dt = m\dot{x}^\star \delta x\Big|_0^T - \int_0^T m\ddot{x}^\star \delta x\, dt.$$

The first term goes to zero because of the boundary condition $\delta x(0) = \delta x(T) = 0$, and the action becomes

$$
\begin{aligned}
S[x] = &\int_0^T (\frac{1}{2}m\dot{x}^{\star 2} - \frac{1}{2}m\omega_0^2 x^{\star 2})\ dt + \int_0^T (\frac{1}{2}m\delta\dot{x}^2 - \frac{1}{2}m\omega_0^2\delta x^2)\ dt \\
&- \int_0^T m(\ddot{x}^\star + \omega_0^2 x^\star)\delta x\ dt.
\end{aligned}
$$

Notice that the last term equals to zero, because its integrand satisfies the equation of motion. Therefore, we obtain the relation $S[x] = S[x^\star] + \delta S$.

(b)

$$
\begin{aligned}
\delta S &= \frac{1}{2}m\Delta^2 \int_0^T \left[(\frac{N\pi}{T})^2 \cos^2(\frac{N\pi t}{T}) - \omega_0^2 \sin^2(\frac{N\pi t}{T}) \right]\ dt \\
&= \frac{1}{2}m\Delta^2 \int_0^T \left[(\frac{N\pi}{T})^2\ \frac{1 + \cos(2N\pi t/T)}{2} - \omega_0^2\ \frac{1 - \cos(2N\pi t/T)}{2} \right]\ dt \\
&= \frac{1}{4}mT\ [(\frac{N\pi}{T})^2 - \omega_0^2]\ \Delta^2 \\
&\equiv c_N \Delta^2.
\end{aligned}
$$

For the case $\omega_0 T > \pi$,

$$
\begin{aligned}
c_N &\equiv \frac{1}{4}mT\ [(\frac{N\pi}{T})^2 - \omega_0^2] \\
&= \frac{1}{4}mT\ (\frac{N\pi}{T} - \omega_0)(\frac{N\pi}{T} + \omega_0).
\end{aligned}
$$

Therefore, the sign of c_N can be either positive of negative, depending on the value of N. This means that δS can be either positive or negative along the special path δx. Hence, the extremal trajectory $x^\star$ given by Hamilton's principle is neither a maximum nor a minimum, but a saddle point.