Assignment 4 solutions

1.

$$\begin{aligned}
\frac{\partial}{\partial q}\frac{df}{dt} &= \frac{\partial}{\partial q}(f_1\,\dot{q} + f_2\,\ddot{q} + f_3) \\
&= \frac{\partial f_1}{\partial q}\dot{q} + \frac{\partial f_2}{\partial q}\ddot{q} + \frac{\partial f_3}{\partial q} \\
&= \frac{\partial f_1}{\partial q}\dot{q} + \frac{\partial}{\partial q}\frac{\partial f}{\partial \dot{q}}\ddot{q} + \frac{\partial}{\partial q}\frac{\partial f}{\partial t} \\
&= \frac{\partial}{\partial q}\frac{\partial f}{\partial q}\dot{q} + \frac{\partial}{\partial \dot{q}}\frac{\partial f}{\partial q}\ddot{q} + \frac{\partial}{\partial t}\frac{\partial f}{\partial q} \\
&= \frac{d}{dt}\frac{\partial f}{\partial q}
\end{aligned}$$

Here we have used the relation

$$\frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial x_2}\frac{\partial f}{\partial x_1}$$

as $f(x_1, x_2)$ is a smooth function.

When replacing q with $\dot{q}$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial \dot{q}}\frac{df}{dt} &= \frac{\partial}{\partial \dot{q}}\left[\frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial \dot{q}}\ddot{q} + \frac{\partial f}{\partial t}\right] \\
&= \frac{\partial^2 f}{\partial \dot{q}\partial q}\dot{q} + \frac{\partial f}{\partial q} + \frac{\partial^2 f}{\partial \dot{q}^2}\ddot{q} + \frac{\partial^2 f}{\partial \dot{q}\partial t}
\end{aligned}$$

and

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{q}} = \frac{\partial^2 f}{\partial q\partial \dot{q}}\dot{q} + \frac{\partial^2 f}{\partial \dot{q}^2}\ddot{q} + \frac{\partial^2 f}{\partial t\partial \dot{q}}\ .$$

Therefore,

$$\left(\frac{\partial}{\partial \dot{q}}\frac{d}{dt} - \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\right) f = \frac{\partial}{\partial q}\, f.$$

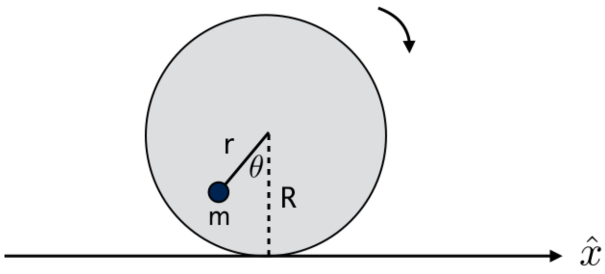
Substituting Lagrangian into the equation above, we have

$$\begin{aligned}
0 &= \left[\frac{\partial}{\partial \dot{q}}\frac{d}{dt} - \frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}\right]\mathcal{L} \\
&= \frac{\partial \dot{\mathcal{L}}}{\partial \dot{q}} - 2\,\frac{\partial \mathcal{L}}{\partial q}.
\end{aligned}$$

Here we have used the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0.$$

2.



Since the wheel rolls without slipping, the position vector of the mass is given by

$$\mathbf{r} = (R\theta - r\sin\theta)\ \hat{x} + (R - r\cos\theta)\ \hat{y}\ .$$

We can hence write the Lagrangian as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\,\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - mgy \\ &= \frac{1}{2}m\left[(R - r\cos\theta)^2 + r^2\sin^2\theta\right]\ \dot{\theta}^2 - mg(R - r\cos\theta) \\ &= \frac{1}{2}m(R^2 + r^2 - 2Rr\cos\theta)\ \dot{\theta}^2 - mg(R - r\cos\theta).\end{aligned}$$

Plugging in the Lagrangian to the equation of motion, we have

$$\begin{aligned}0 &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} \\ &= \frac{d}{dt}\left[m\dot{\theta}(R^2 + r^2 - 2Rr\cos\theta)\right] - m\dot{\theta}^2 Rr\sin\theta + mgr\sin\theta \\ &= m\ddot{\theta}(R^2 + r^2 - 2Rr\cos\theta) + m\dot{\theta}^2 Rr\sin\theta + mgr\sin\theta.\end{aligned}$$

For small θ, we can approximate the equation of motion to

$$\ddot{\theta}(R^2 + r^2 - 2Rr) = -gr\theta\,,$$

which can be viewed as a simple harmonic oscillator with frequency

$$\omega = \frac{\sqrt{gr}}{R - r}\ .$$