Assignment 3 solutions

1. Without torques acting on the rigid body, we can use Euler's rigid body equations

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$$

to express the time evolution of its angular velocity components, and the time evolution of the body basis vectors is given by

$$\dot{\hat{1}} = \vec{\omega} \times \hat{1},\ \dot{\hat{2}} = \vec{\omega} \times \hat{2},\ \dot{\hat{3}} = \vec{\omega} \times \hat{3}\,,$$

where $\vec{\omega} = \omega_1\ \hat{1} + \omega_2\ \hat{2} + \omega_3\ \hat{3}$.

The finite-difference integration of the relations above can be written as

$$\omega_1(t + \Delta t) = \omega_1(t) + \Delta t\ (I_2 - I_3)\ \omega_2(t)\ \omega_3(t)/I_1$$
$$\omega_2(t + \Delta t) = \omega_2(t) + \Delta t\ (I_3 - I_1)\ \omega_3(t)\ \omega_1(t)/I_2$$
$$\omega_3(t + \Delta t) = \omega_3(t) + \Delta t\ (I_1 - I_2)\ \omega_1(t)\ \omega_2(t)/I_3$$

and

$$\hat{1}(t + \Delta t) = \hat{1}(t) + \Delta t\ (\vec{\omega}(t) \times \hat{1}(t)) = \hat{1}(t) + \Delta t\ (\omega_3(t)\ \hat{2}(t) - \omega_2(t)\ \hat{3}(t))$$
$$\hat{2}(t + \Delta t) = \hat{2}(t) + \Delta t\ (\vec{\omega}(t) \times \hat{2}(t)) = \hat{2}(t) + \Delta t\ (\omega_1(t)\ \hat{3}(t) - \omega_3(t)\ \hat{1}(t))$$
$$\hat{3}(t + \Delta t) = \hat{3}(t) + \Delta t\ (\vec{\omega}(t) \times \hat{3}(t)) = \hat{3}(t) + \Delta t\ (\omega_2(t)\ \hat{1}(t) - \omega_1(t)\ \hat{2}(t))\ .$$

Using Python as an example, we can time-evolve the system for one orbit:

```
import numpy as np
I1, I2, I3 = 1, 2, 3
one = np.array([1, 0, 0])
two = np.array([0, 1, 0])
three = np.array([0, 0, 1])
omega0 = np.array([1., 1., 0.9])
omega = np.copy(omega0)

dt = 0.001
diff = 0.
T = 0.
```

```
# exit the while loop when diff < 0.01 and T > 1.
while (diff >= 0.01 or T <= 1.):
   w1, w2, w3 = omega[0], omega[1], omega[2]
   new_w1 = w1 + dt*(I2-I3)*w2*w3/I1
   new_w2 = w2 + dt*(I3-I1)*w3*w1/I2
   new_w3 = w3 + dt*(I1-I2)*w1*w2/I3

   new_one = one + dt*(w3*two - w2*three)
   new_two = two + dt*(w1*three - w3*one)
   new_three = three + dt*(w2*one - w1*two)

   omega = np.array([new_w1, new_w2, new_w3])
   one = np.copy(new_one)
   two = np.copy(new_two)
   three = np.copy(new_three)

   diff = 0.
   for i in xrange(3):
     diff += (omega[i]-omega0[i])**2
   diff = np.sqrt(diff)
   T += dt

U = np.zeros((3, 3))
U[0] = np.copy(one)
U[1] = np.copy(two)
U[2] = np.copy(three)
trace = U[0][0] + U[1][1] + U[2][2]
theta = np.arccos((trace-1)/2)

print T
print U
print np.dot(U, U.T)
print theta
```

The program gives us

$$
\begin{aligned}
T &= 7.22\,,\\
U &= \begin{bmatrix} -0.682 & -0.141 & 0.730 \\ 0.739 & -0.239 & 0.642 \\ 0.080 & 0.967 & 0.257 \end{bmatrix},\\
UU^T &= \begin{bmatrix} 1.018 & -0.002 & -0.003 \\ -0.002 & 1.015 & -0.007 \\ -0.003 & -0.007 & 1.008 \end{bmatrix},\\
\theta &= 2.55 \text{ rad}\,.
\end{aligned}
$$

2. Assume that the ladder has length L and mass M. The kinetic energy of the ladder can be separated into the translational part T_{trans} and rotational part T_{rot}. As the wall has position $w(t)$, the position vector of the center of mass of the ladder can be expressed by

$$\mathbf{r} = (w(t) + \frac{L}{2}\sin\theta)\ \hat{x} + \frac{L}{2}\cos\theta\ \hat{y}\ .$$

Taking the time derivative, we obtain the velocity

$$\dot{\mathbf{r}} = (\dot{w}(t) + \frac{L}{2}\dot{\theta}\cos\theta)\ \hat{x} - \frac{L}{2}\dot{\theta}\sin\theta\ \hat{y}\ .$$

The translational kinetic energy is then given by

$$T_{trans} = \frac{M}{2}\ \dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \frac{M}{2}\ \left[\dot{w}(t)^2 + \dot{w}(t)\dot{\theta}\ L\cos\theta + \frac{L^2}{4}\ \dot{\theta}^2\right]\ .$$

The rotational kinetic energy is given by

$$T_{rot} = \frac{1}{2}\ I\dot{\theta}^2 = \frac{1}{2}\ \frac{ML^2}{12}\ \dot{\theta}^2\ .$$

Therefore, the total kinetic energy

$$T = T_{trans} + T_{rot} = \frac{M}{2}\ \left[\dot{w}(t)^2 + \dot{w}(t)\dot{\theta}\ L\cos\theta + \frac{L^2}{3}\ \dot{\theta}^2\right]\ .$$

The potential energy V can be written as

$$V = Mg\ \frac{L}{2}\cos\theta\,,$$

so the Lagrangian is given by

$$\mathcal{L} = T - V = \frac{M}{2}\ \left[\dot{w}(t)^2 + (\dot{w}(t)\dot{\theta} - g)\ L\cos\theta + \frac{L^2}{3}\ \dot{\theta}^2\right]\ .$$

Taking derivative with respect to θ and $\dot{\theta}$ respectively,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\theta} &= -\frac{M}{2}\ (\dot{w}(t)\dot{\theta} - g)\ L\sin\theta \\ \frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= \frac{M}{2}\ \left[\dot{w}(t)\ L\cos\theta + \frac{2}{3}\ L^2\dot{\theta}\right]\,,\end{aligned}$$

and the equation of motion

$$\begin{aligned}0 &= \frac{d}{dt}\ \frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} \\ &= \frac{M}{2}\ \left[\ddot{w}(t)L\cos\theta - gL\sin\theta + \frac{2}{3}\ L^2\ \ddot{\theta}\right]\ .\end{aligned}$$

For the special solution that θ remains constant, the equation of motion becomes

$$0 = \frac{M}{2}\, L\, (\ddot{w}(t)\cos\theta - g\sin\theta)\,,$$

so the position of the wall must satisfy

$$\ddot{w}(t) = g\tan\theta\,,$$

or

$$w(t) = \frac{1}{2}\, g\tan\theta\, t^2 + at + b\,,$$

where a and b are constants determined by the initial conditions.