Assignment 3 solutions

1. Without torques acting on the rigid body, we can use Euler’s rigid body equations

\[
I_1 \dot{\omega}_1 = (I_2 - I_3) \omega_2 \omega_3 \\
I_2 \dot{\omega}_2 = (I_3 - I_1) \omega_3 \omega_1 \\
I_3 \dot{\omega}_3 = (I_1 - I_2) \omega_1 \omega_2
\]

to express the time evolution of its angular velocity components, and the time evolution of the body basis vectors is given by

\[
\dot{\hat{1}} = \vec{\omega} \times \hat{1}, \quad \dot{\hat{2}} = \vec{\omega} \times \hat{2}, \quad \dot{\hat{3}} = \vec{\omega} \times \hat{3},
\]

where \( \vec{\omega} = \omega_1 \hat{1} + \omega_2 \hat{2} + \omega_3 \hat{3} \).

The finite-difference integration of the relations above can be written as

\[
\omega_1(t + \Delta t) = \omega_1(t) + \Delta t \left( I_2 - I_3 \right) \omega_2(t) \omega_3(t)/I_1 \\
\omega_2(t + \Delta t) = \omega_2(t) + \Delta t \left( I_3 - I_1 \right) \omega_3(t) \omega_1(t)/I_2 \\
\omega_3(t + \Delta t) = \omega_3(t) + \Delta t \left( I_1 - I_2 \right) \omega_1(t) \omega_2(t)/I_3
\]

and

\[
\dot{\hat{1}}(t + \Delta t) = \dot{\hat{1}}(t) + \Delta t (\vec{\omega}(t) \times \hat{1}(t)) = \dot{\hat{1}}(t) + \Delta t (\omega_3(t) \hat{2}(t) - \omega_2(t) \hat{3}(t)) \\
\dot{\hat{2}}(t + \Delta t) = \dot{\hat{2}}(t) + \Delta t (\vec{\omega}(t) \times \hat{2}(t)) = \dot{\hat{2}}(t) + \Delta t (\omega_1(t) \hat{3}(t) - \omega_3(t) \hat{1}(t)) \\
\dot{\hat{3}}(t + \Delta t) = \dot{\hat{3}}(t) + \Delta t (\vec{\omega}(t) \times \hat{3}(t)) = \dot{\hat{3}}(t) + \Delta t (\omega_2(t) \hat{1}(t) - \omega_1(t) \hat{2}(t)).
\]

Using Python as an example, we can time-evolve the system for one orbit:

```python
import numpy as np
I1, I2, I3 = 1, 2, 3
one = np.array([1, 0, 0])
two = np.array([0, 1, 0])
three = np.array([0, 0, 1])
omega0 = np.array([1., 1., 0.9])
omega = np.copy(omega0)
dt = 0.001
diff = 0.
T = 0.
```
# exit the while loop when diff < 0.01 and T > 1.
while (diff >= 0.01 or T <= 1.):
    w1, w2, w3 = omega[0], omega[1], omega[2]
    new_w1 = w1 + dt*(I2-I3)*w2*w3/I1
    new_w2 = w2 + dt*(I3-I1)*w3*w1/I2
    new_w3 = w3 + dt*(I1-I2)*w1*w2/I3

    new_one = one + dt*(w3*two - w2*three)
    new_two = two + dt*(w1*three - w3*one)
    new_three = three + dt*(w2*one - w1*two)

    omega = np.array([new_w1, new_w2, new_w3])
    one = np.copy(new_one)
    two = np.copy(new_two)
    three = np.copy(new_three)

    diff = 0.
    for i in xrange(3):
        diff += (omega[i]-omega0[i])**2
    diff = np.sqrt(diff)
    T += dt

U = np.zeros((3, 3))
U[0] = np.copy(one)
U[1] = np.copy(two)
U[2] = np.copy(three)
trace = U[0][0] + U[1][1] + U[2][2]
theta = np.arccos((trace-1)/2)

print T
print U
print np.dot(U, U.T)
print theta

The program gives us

\[ T = 7.22, \]
\[
U = \begin{bmatrix}
-0.682 & -0.141 & 0.730 \\
0.739 & -0.239 & 0.642 \\
0.080 & 0.967 & 0.257
\end{bmatrix},
\]
\[
UU^T = \begin{bmatrix}
1.018 & -0.002 & -0.003 \\
-0.002 & 1.015 & -0.007 \\
-0.003 & -0.007 & 1.008
\end{bmatrix},
\]
\[ \theta = 2.55 \text{ rad}. \]
2. Assume that the ladder has length $L$ and mass $M$. The kinetic energy of the ladder can be separated into the translational part $T_{\text{trans}}$ and rotational part $T_{\text{rot}}$. As the wall has position $w(t)$, the position vector of the center of mass of the ladder can be expressed by

$$
r = (w(t) + \frac{L}{2} \sin \theta) \hat{x} + \frac{L}{2} \cos \theta \hat{y}.
$$

Taking the time derivative, we obtain the velocity

$$
\dot{r} = (\dot{w}(t) + \frac{L}{2} \dot{\theta} \cos \theta) \hat{x} - \frac{L}{2} \dot{\theta} \sin \theta \hat{y}.
$$

The translational kinetic energy is then given by

$$
T_{\text{trans}} = \frac{M}{2} \dot{r} \cdot \dot{r} = \frac{M}{2} \left[ \dot{w}(t)^2 + 2 \dot{w}(t) \dot{\theta} L \cos \theta + \frac{L^2}{4} \dot{\theta}^2 \right].
$$

The rotational kinetic energy is given by

$$
T_{\text{rot}} = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} \frac{ML^2}{12} \dot{\theta}^2.
$$

Therefore, the total kinetic energy

$$
T = T_{\text{trans}} + T_{\text{rot}} = \frac{M}{2} \left[ \dot{w}(t)^2 + 2 \dot{w}(t) \dot{\theta} L \cos \theta + \frac{L^2}{3} \dot{\theta}^2 \right].
$$

The potential energy $V$ can be written as

$$
V = Mg \frac{L}{2} \cos \theta,
$$

so the Lagrangian is given by

$$
\mathcal{L} = T - V = \frac{M}{2} \left[ \ddot{w}(t)^2 + (\dot{w}(t) \dot{\theta} - g) L \cos \theta + \frac{L^2}{3} \dot{\theta}^2 \right].
$$

Taking derivative with respect to $\theta$ and $\dot{\theta}$ respectively,

$$
\frac{\partial \mathcal{L}}{\partial \theta} = -\frac{M}{2} (\dot{w}(t) \dot{\theta} - g) L \sin \theta,
$$

$$
\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{M}{2} \left[ \ddot{w}(t) L \cos \theta + \frac{2}{3} L^2 \ddot{\theta} \right],
$$

and the equation of motion

$$
0 = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{M}{2} \left[ \ddot{w}(t) L \cos \theta - gL \sin \theta + \frac{2}{3} L^2 \ddot{\theta} \right].
$$
For the special solution that $\theta$ remains constant, the equation of motion becomes

$$0 = \frac{M}{2} L (\ddot{w}(t) \cos \theta - g \sin \theta),$$

so the position of the wall must satisfy

$$\ddot{w}(t) = g \tan \theta,$$

or

$$w(t) = \frac{1}{2} g \tan \theta t^2 + at + b,$$

where $a$ and $b$ are constants determined by the initial conditions.