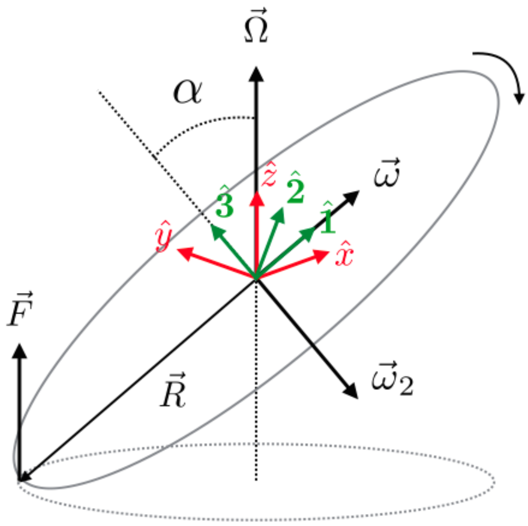
Assignment 2 solutions

1.



Recall that the total angular velocity $\vec{\omega}$ of the ring that rolls clockwise without slipping is anti-parallel to the position vector $\vec{R}$ of the point of contact. Consider the body frame S' with its basis vector $\hat{\mathbf{3}}$ parallel to the axis of the ring and $\hat{\mathbf{1}}$ always aligning with $\vec{\omega}$. Assume that the ring has linear mass density $\lambda = M/(2\pi R)$. The summation of the formula (4.17) in Lecture 4 becomes an integral:

$$\begin{aligned}\mathbf{I}' &= \int dm\ ((\mathbf{r}\cdot\mathbf{r})\mathbb{1} - \mathbf{rr}) \\ &= \lambda R \int_0^{2\pi} d\theta\ ((\mathbf{r}\cdot\mathbf{r})\mathbb{1} - \mathbf{rr}),\end{aligned}$$

where

$$\int_0^{2\pi} d\theta\ (\mathbf{r}\cdot\mathbf{r})\mathbb{1} = \int_0^{2\pi} d\theta\ R^2\ \mathbb{1} = 2\pi R^2(\hat{\mathbf{1}}\hat{\mathbf{1}} + \hat{\mathbf{2}}\hat{\mathbf{2}} + \hat{\mathbf{3}}\hat{\mathbf{3}})$$

$$\begin{aligned}\int_0^{2\pi} d\theta\ (\mathbf{rr}) &= R^2 \int_0^{2\pi} d\theta\ (\cos\theta\ \hat{\mathbf{1}} + \sin\theta\ \hat{\mathbf{2}})(\cos\theta\ \hat{\mathbf{1}} + \sin\theta\ \hat{\mathbf{2}}) \\ &= R^2 \int_0^{2\pi} d\theta\ (\cos^2\theta\ \hat{\mathbf{1}}\hat{\mathbf{1}} + \cos\theta\sin\theta\ (\hat{\mathbf{1}}\hat{\mathbf{2}} + \hat{\mathbf{2}}\hat{\mathbf{1}}) + \sin^2\theta\ \hat{\mathbf{2}}\hat{\mathbf{2}}) \\ &= \pi R^2\ (\hat{\mathbf{1}}\hat{\mathbf{1}} + \hat{\mathbf{2}}\hat{\mathbf{2}}).\end{aligned}$$

We obtain

$$\mathbf{I}' = \frac{MR^2}{2}(\hat{\mathbf{1}}\hat{\mathbf{1}} + \hat{\mathbf{2}}\hat{\mathbf{2}} + 2\ \hat{\mathbf{3}}\hat{\mathbf{3}}).$$

In the body frame S', the total angular momentum has the form $\vec{\omega}' = \omega' \, \hat{\mathbf{1}}$, so the angular momentum $\mathbf{L}'$ of the ring in S' can be expressed as

$$\mathbf{L}' = \mathbf{I}' \cdot \vec{\omega}' = \frac{MR^2}{2} \, \omega' \, \hat{\mathbf{1}} = \frac{MR^2}{2} \, \vec{\omega}'$$

Applying orthogonal transformation back to the space frame S, we have

$$\mathbf{L} = \frac{MR^2}{2} \, \vec{\omega}.$$

From the first assignment, the total angular momentum $\vec{\omega}$ in the space frame S can be expressed as

$$\vec{\omega}(t) = \Omega \sin\alpha \, [\cos\alpha \cos(\Omega t) \, \hat{x} + \cos\alpha \sin(\Omega t) \, \hat{y} + \sin\alpha \, \hat{z}] \, ,$$

if we assume that $\vec{\omega}$ lies on xz plane at $t = 0$. Hence, the angular momentum $\mathbf{L}$ becomes

$$\mathbf{L}(t) = \frac{MR^2}{2} \, \Omega \sin\alpha \, [\cos\alpha \cos(\Omega t) \, \hat{x} + \cos\alpha \sin(\Omega t) \, \hat{y} + \sin\alpha \, \hat{z}] \, ,$$

and the torque $\mathbf{N}$ is given by

$$\begin{aligned} \mathbf{N}(t) = \dot{\mathbf{L}}(t) &= \frac{MR^2}{2} \, \Omega^2 \cos\alpha \sin\alpha (-\sin(\Omega t) \, \hat{x} + \cos(\Omega t) \, \hat{y}) \\ &= \frac{MR^2}{2} \, \Omega \, \hat{z} \times \vec{\omega}. \end{aligned}$$

Because the center of mass of the ring stays fixed, we only expect a normal force $\vec{F} = Mg \, \hat{z}$ at the point of contact. We thus have

$$\begin{aligned} \mathbf{N}(t) &= \vec{R}(t) \times \vec{F} = (-R \, \frac{\vec{\omega}}{\omega}) \times (Mg \, \hat{z}) = \frac{MgR}{\omega} \, \hat{z} \times \vec{\omega} = \frac{MgR}{\Omega \sin\alpha} \, \hat{z} \times \vec{\omega} \\ &= \frac{MR^2}{2} \, \Omega \, \hat{z} \times \vec{\omega}. \end{aligned}$$

The angular velocity Ω is then given by

$$\Omega = \sqrt{\frac{2g}{R \sin\alpha}}.$$

At tilt angle α, the potential energy $V(\alpha)$ of the system is given by

$$V(\alpha) = MgR \sin\alpha.$$

Because the center of mass of the ring stays fixed, only the rotational part contribute to the kinetic energy.

$$\begin{aligned} T(\alpha) &= \frac{1}{2} \, \vec{\omega} \cdot \mathbf{I} \cdot \vec{\omega} \\ &= \frac{1}{2} \, \vec{\omega} \cdot \mathbf{L} = \frac{1}{4} \, MR^2 \omega^2 = \frac{1}{4} \, MR^2 \Omega^2 \sin^2\alpha = \frac{1}{2} \, MgR \sin\alpha. \end{aligned}$$

Therefore, we have the total energy $E(\alpha)$

$$E(\alpha) = V(\alpha) + T(\alpha) = \frac{3}{2}\, MgR \sin\alpha,$$

and we can express the angular velocity Ω as a function of E by

$$\Omega(E) = \sqrt{\frac{3Mg^2}{E}}.$$

Suppose friction decreases the total energy E linearly to zero with time at the rate

$$E(t) = E(0) - kt$$

the angular velocity Ω will then increase at the rate

$$\Omega(t) = \sqrt{\frac{3Mg^2}{E(0) - kt}}.$$

2. The angular momentum $\mathbf{L}$ is given by

$$\begin{aligned}
\mathbf{L} &= \sum_i (\mathbf{R} + \mathbf{r}_i) \times (m_i \dot{\mathbf{r}}_i) \\
&= \sum_i m_i \; (\mathbf{R} + \mathbf{r}_i) \times (\dot{\mathbf{R}} + \vec{\omega} \times \mathbf{r}_i) \\
&= (\sum_i m_i)\; \mathbf{R} \times \dot{\mathbf{R}} + (\sum_i m_i \mathbf{r}_i) \times \dot{\mathbf{R}} + \mathbf{R} \times (\vec{\omega} \times \sum_i m_i \mathbf{r}_i) + \sum_i m_i \mathbf{r}_i \times (\vec{\omega} \times \mathbf{r}_i) \\
&= M\; \mathbf{R} \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{r}_i \times (\vec{\omega} \times \mathbf{r}_i).
\end{aligned}$$

The cross terms go to zero because of the center of mass property

$$0 = \sum_i m_i \mathbf{r}_i.$$

When $\mathbf{R}$ is at rest, $\dot{\mathbf{R}} = 0$, and

$$\begin{aligned}
\mathbf{L} &= \sum_i m_i \mathbf{r}_i \times (\vec{\omega} \times \mathbf{r}_i) \\
&= \sum_i m_i (\vec{\omega}(\mathbf{r}_i \cdot \mathbf{r}_i) - \mathbf{r}_i(\mathbf{r}_i \cdot \vec{\omega})) \\
&= \sum_i m_i ((\mathbf{r}_i \cdot \mathbf{r}_i)\mathbb{1} - \mathbf{r}_i \mathbf{r}_i) \cdot \vec{\omega} = \mathbf{I} \cdot \vec{\omega}.
\end{aligned}$$

3. Applying time derivative to

$$\mathbf{I} = I_1\; \hat{\mathbf{1}}\hat{\mathbf{1}} + I_2\; \hat{\mathbf{2}}\hat{\mathbf{2}} + I_3\; \hat{\mathbf{3}}\hat{\mathbf{3}},$$

we have

$$\dot{\mathbf{I}} = I_1\ (\dot{\hat{\mathbf{1}}}\hat{\mathbf{1}} + \hat{\mathbf{1}}\dot{\hat{\mathbf{1}}}) + I_2\ (\dot{\hat{\mathbf{2}}}\hat{\mathbf{2}} + \hat{\mathbf{2}}\dot{\hat{\mathbf{2}}}) + I_3\ (\dot{\hat{\mathbf{3}}}\hat{\mathbf{3}} + \hat{\mathbf{3}}\dot{\hat{\mathbf{3}}})$$

from the fact that the body-fixed principal moments of inertia are time independent.

Recall that the time evolution in space frame of an arbitrary point $\mathbf{r}$ of a rigid body obeys the rule

$$\dot{\mathbf{r}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = A\mathbf{r} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{\omega} \times \mathbf{r},$$

so

$$\dot{\hat{\mathbf{1}}}\hat{\mathbf{1}} + \hat{\mathbf{1}}\dot{\hat{\mathbf{1}}} = (\vec{\omega} \times \hat{\mathbf{1}})\hat{\mathbf{1}} + \hat{\mathbf{1}}(\vec{\omega} \times \hat{\mathbf{1}}) = (\vec{\omega} \times \hat{\mathbf{1}})\hat{\mathbf{1}} - \hat{\mathbf{1}}(\hat{\mathbf{1}} \times \vec{\omega}).$$

We can hence simplify $\dot{\mathbf{I}}$ as

$$\begin{aligned} \dot{\mathbf{I}} &= I_1\ [(\vec{\omega} \times \hat{\mathbf{1}})\hat{\mathbf{1}} - \hat{\mathbf{1}}(\hat{\mathbf{1}} \times \vec{\omega})] + I_2\ [(\vec{\omega} \times \hat{\mathbf{2}})\hat{\mathbf{2}} - \hat{\mathbf{2}}(\hat{\mathbf{2}} \times \vec{\omega})] \\ &\quad + I_3\ [(\vec{\omega} \times \hat{\mathbf{3}})\hat{\mathbf{3}} - \hat{\mathbf{3}}(\hat{\mathbf{3}} \times \vec{\omega})] \\ &= \vec{\omega} \times \mathbf{I} - \mathbf{I} \times \vec{\omega}. \end{aligned} \tag{1}$$

Substituting

$$\vec{\omega} = \omega_1\ \hat{\mathbf{1}} + \omega_2\ \hat{\mathbf{2}} + \omega_3\ \hat{\mathbf{3}},$$

into Eq. (1), we obtain

$$\dot{\mathbf{I}} = \begin{bmatrix} 0 & \omega_3(I_1 - I_2) & \omega_2(I_3 - I_1) \\ \omega_3(I_1 - I_2) & 0 & \omega_1(I_2 - I_3) \\ \omega_2(I_3 - I_1) & \omega_1(I_2 - I_3) & 0 \end{bmatrix}.$$

- $I_1 < I_2 < I_3$:
  $\dot{\mathbf{I}} = 0$ when $\omega_1 = \omega_2 = \omega_3 = 0$.
- $I_1 = I_2 = I_3$:
  $\dot{\mathbf{I}} = 0$ for arbitrary values of $(\omega_1,\ \omega_2,\ \omega_3)$.
- $I_1 = I_2 < I_3$:

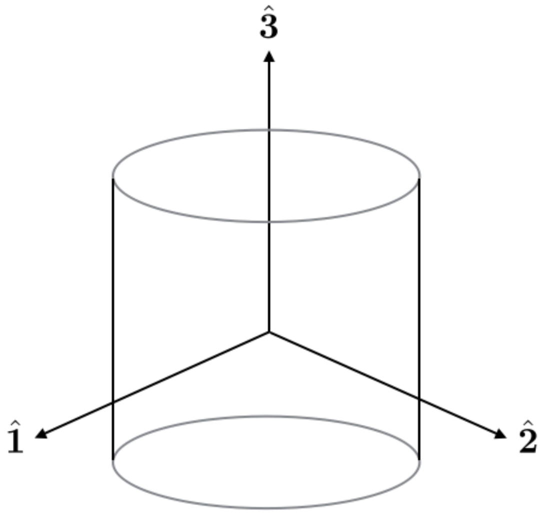
  $$\dot{\mathbf{I}} = \begin{bmatrix} 0 & 0 & \omega_2(I_3 - I_1) \\ 0 & 0 & \omega_1(I_2 - I_3) \\ \omega_2(I_3 - I_1) & \omega_1(I_2 - I_3) & 0 \end{bmatrix},$$

  and $\dot{\mathbf{I}} = 0$ when $\omega_1 = \omega_2 = 0$, where the symmetric top spins about its own axis at the angular velocity ω_3 without precession.

4.

Consider the body frame with its origin at the center of mass of the cylinder, and its basis vector $\hat{\mathbf{3}}$ parallel to the axis of the cylinder. The moment of inertia tensor $\mathbf{I}$ is given by

$$\mathbf{I} = \int dm\ ((\mathbf{r} \cdot \mathbf{r})\mathbb{I} - \mathbf{r}\mathbf{r}).$$



For an arbitrary point $\mathbf{r} = r' \cos\phi\, \hat{\mathbf{1}} + r' \sin\phi\, \hat{\mathbf{2}} + z\, \hat{\mathbf{3}}$,

$$\begin{aligned}
\int dm\, (\mathbf{r}\cdot\mathbf{r})\mathbb{1} &= \int_{-h/2}^{h/2}\int_0^r \int_0^{2\pi} \rho\, r' dr'\, d\phi\, dz\, (r'^2 + z^2)\mathbb{1} \\
&= \frac{\pi}{2}\, \rho\, (r^4 h + \frac{1}{6}\, r^2 h^3)\mathbb{1} \\
\int dm\, \mathbf{rr} &= \int_{-h/2}^{h/2}\int_0^r \int_0^{2\pi} \rho\, r' dr'\, d\phi\, dz\, \Big[r'^2 \cos^2\phi\, \hat{\mathbf{1}}\hat{\mathbf{1}} + r'^2 \sin^2\phi\, \hat{\mathbf{2}}\hat{\mathbf{2}} + z^2\, \hat{\mathbf{3}}\hat{\mathbf{3}} \\
&\quad + r'^2 \cos\phi \sin\phi\, (\hat{\mathbf{1}}\hat{\mathbf{2}} + \hat{\mathbf{2}}\hat{\mathbf{1}}) + r' z \cos\phi\, (\hat{\mathbf{1}}\hat{\mathbf{3}} + \hat{\mathbf{3}}\hat{\mathbf{1}}) + r' z \sin\phi\, (\hat{\mathbf{2}}\hat{\mathbf{3}} + \hat{\mathbf{3}}\hat{\mathbf{2}}) \Big] \\
&= \int_{-h/2}^{h/2}\int_0^r \int_0^{2\pi} \rho\, r' dr'\, d\phi\, dz\, \Big[r'^2 \cos^2\phi\, \hat{\mathbf{1}}\hat{\mathbf{1}} + r'^2 \sin^2\phi\, \hat{\mathbf{2}}\hat{\mathbf{2}} + z^2\, \hat{\mathbf{3}}\hat{\mathbf{3}} \Big] \\
&= \frac{\pi}{4}\, \rho r^4 h\, (\hat{\mathbf{1}}\hat{\mathbf{1}} + \hat{\mathbf{2}}\hat{\mathbf{2}}) + \frac{\pi}{12}\, \rho r^2 h^3\, \hat{\mathbf{3}}\hat{\mathbf{3}}.
\end{aligned}$$

Note that

$$\int_0^{2\pi} d\phi \cos\phi = \int_0^{2\pi} d\phi \sin\phi = \int_0^{2\pi} d\phi \cos\phi \sin\phi = 0.$$

We obtain the moment of inertia tensor $\mathbf{I}$ as

$$\mathbf{I} = \frac{\pi}{4}\, \rho\, (r^4 h + \frac{1}{3}\, r^2 h^3)\, (\hat{\mathbf{1}}\hat{\mathbf{1}} + \hat{\mathbf{2}}\hat{\mathbf{2}}) + \frac{\pi}{2}\, \rho\, r^4 h\, \hat{\mathbf{3}}\hat{\mathbf{3}}.$$

When $h/r = \sqrt{3}$,

$$\mathbf{I} = \frac{\pi}{18}\, \rho h^5\, (\hat{\mathbf{1}}\hat{\mathbf{1}} + \hat{\mathbf{2}}\hat{\mathbf{2}} + \hat{\mathbf{3}}\hat{\mathbf{3}}),$$

whose rotational behavior is indistinguishable from a sphere.