Assignment 13 solutions

1. The Lagrangian of a relativistic string is given by

$$L = \sqrt{-a^{\alpha\beta}a_{\alpha\beta}}\,,$$

where

$$a^{\alpha\beta} = \frac{\epsilon^{ab}}{\sqrt{2}}\,\partial_a s^\alpha \partial_b s^\beta.$$

To simplify the Euler-Lagrange equation:

$$0 = \partial_p\left(\frac{\partial L}{\partial(\partial_p s^\alpha)}\right) + \partial_q\left(\frac{\partial L}{\partial(\partial_q s^\alpha)}\right) = \partial_a\left(\frac{\partial L}{\partial(\partial_a s^\alpha)}\right),$$

we start with the argument of the partial derivative:

$$\begin{aligned}
\frac{\partial L}{\partial(\partial_a s^\alpha)} &= -\frac{1}{2L}\left(a_{\beta\gamma}\,\frac{\partial a^{\beta\gamma}}{\partial(\partial_a s^\alpha)} + a^{\beta\gamma}\,\frac{\partial a_{\beta\gamma}}{\partial(\partial_a s^\alpha)}\right)\\
&= -\frac{1}{2L}\left(a_{\beta\gamma}\,\frac{\partial a^{\beta\gamma}}{\partial(\partial_a s^\alpha)} + a_{\beta'\gamma'}g^{\beta'\beta}g^{\gamma'\gamma}g_{\beta\mu}g_{\gamma\nu}\,\frac{\partial a^{\mu\nu}}{\partial(\partial_a s^\alpha)}\right)\\
&= -\frac{1}{2L}\left(a_{\beta\gamma}\,\frac{\partial a^{\beta\gamma}}{\partial(\partial_a s^\alpha)} + a_{\beta'\gamma'}\delta^{\beta'}{}_{\mu}\delta^{\gamma'}{}_{\nu}\,\frac{\partial a^{\mu\nu}}{\partial(\partial_a s^\alpha)}\right)\\
&= -\frac{1}{2L}\left(a_{\beta\gamma}\,\frac{\partial a^{\beta\gamma}}{\partial(\partial_a s^\alpha)} + a_{\mu\nu}\,\frac{\partial a^{\mu\nu}}{\partial(\partial_a s^\alpha)}\right)\\
&= -\frac{1}{L}\left(a_{\beta\gamma}\,\frac{\partial a^{\beta\gamma}}{\partial(\partial_a s^\alpha)}\right)\\
&= -\frac{\epsilon^{bc}}{\sqrt{2}L}\,a_{\beta\gamma}\,\frac{\partial}{\partial(\partial_a s^\alpha)}\left(\partial_b s^\beta \partial_c s^\gamma\right)\\
&= -\frac{\epsilon^{bc}}{\sqrt{2}L}\,a_{\beta\gamma}\left(\delta_b{}^a\delta_\alpha{}^\beta\partial_c s^\gamma + \delta_c{}^a\delta_\alpha{}^\gamma\partial_b s^\beta\right)\\
&= -\frac{1}{\sqrt{2}L}\,\left(\epsilon^{ac}a_{\alpha\gamma}\partial_c s^\gamma + \epsilon^{ba}a_{\beta\alpha}\partial_b s^\beta\right)\\
&= -\frac{1}{\sqrt{2}L}\,\left(\epsilon^{ac}a_{\alpha\gamma}\partial_c s^\gamma + \epsilon^{ab}a_{\alpha\beta}\partial_b s^\beta\right)\\
&= -\frac{\sqrt{2}}{L}\,\epsilon^{ab}a_{\alpha\beta}\partial_b s^\beta.
\end{aligned}$$

In the second last line above, we swap the indices of ϵ^{ab} and $a_{\alpha\beta}$ at the same time because they are both antisymmetric.

Therefore,

$$
\begin{aligned}
0 &= \partial_a \left(\frac{\partial L}{\partial(\partial_a s^\alpha)} \right) \\
&= -\sqrt{2}\, \partial_a \left(\epsilon^{ab}\, \frac{a_{\alpha\beta}}{L}\, \partial_b s^\beta \right) \\
&= -\sqrt{2}\, \epsilon^{ab} \left(\partial_a (\frac{a_{\alpha\beta}}{L})\, \partial_b s^\beta + \frac{a_{\alpha\beta}}{L}\, \partial_{ab} s^\beta \right) \\
&= -\sqrt{2}\, \epsilon^{ab} \left(\partial_a (\frac{a_{\alpha\beta}}{L})\, \partial_b s^\beta \right)
\end{aligned}
$$

because $\partial_{ab} s^\beta = \partial_{ba} s^\beta$.

We can further rewrite the above equation as

$$
\begin{aligned}
0 &= \epsilon^{ab} \left(\partial_a (\frac{a_{\alpha\beta}}{L})\, \partial_b s^\beta \right) \\
&= \epsilon^{ab} \left(\partial_a (\frac{a^{\mu\nu}}{L}) g_{\mu\alpha} g_{\nu\beta}\, \partial_b s^\beta \right) \\
&= g_{\mu\alpha}\, \epsilon^{ab} \partial_a (\frac{a^{\mu\nu}}{L})\, (\partial_b s_\nu).
\end{aligned}
$$

Since ν is only a dummy variable in summation, we can change ν to β. Moreover, $g_{\mu\alpha}$ is nonzero only when $\alpha = \mu$. We thus obtain the equation of motion for relativistic strings:

$$
0 = \epsilon^{ab} (\partial_a v^{\alpha\beta})(\partial_b s_\beta).
$$

2. With the world surface specified by

$$
s^\alpha(x,t) = (ct,\ x,\ y(x,t),\ 0),
$$

we can readily get its two tangent vectors

$$
\begin{aligned}
\partial_x s^\alpha &= (0,\ 1,\ \partial_x y,\ 0) \\
\partial_t s^\alpha &= (c,\ 0,\ \partial_t y,\ 0).
\end{aligned}
$$

From these we can calculate the square of the Lagrangian:

$$
\begin{aligned}
L^2 &= -a^{\alpha\beta} a_{\alpha\beta} \\
&= (\partial_x s^\alpha \partial_t s_\alpha)^2 - (\partial_x s^\alpha \partial_x s_\alpha)(\partial_t s^\alpha \partial_t s_\alpha) \\
&= (\partial_x y)^2 (\partial_t y)^2 - ((\partial_x y)^2 + 1)((\partial_t y)^2 - c^2) \\
&= c^2 (1 + (\partial_x y)^2) - (\partial_t y)^2,
\end{aligned}
$$

and the action:

$$
\begin{aligned}
S[y] &= \int \sqrt{c^2 (1 + (\partial_x y)^2) - (\partial_t y)^2}\, dx dt \\
&= \int L(\partial_x y,\ \partial_t y)\, dx dt.
\end{aligned}
$$

Hereafter, we will set $c = 1$ by suitably choosing the units. Substituting $L(\partial_x y,\ \partial_t y)$ into the two-variable Euler-Lagrange equation:

$$0 = \frac{\partial L}{\partial y} - \partial_x \left(\frac{\partial L}{\partial(\partial_x y)} \right) - \partial_t \left(\frac{\partial L}{\partial(\partial_t y)} \right),$$

we have

$$\begin{aligned}
0 &= \partial_x \left(\frac{\partial L}{\partial(\partial_x y)} \right) + \partial_t \left(\frac{\partial L}{\partial(\partial_t y)} \right) \\
&= \partial_x \left(\frac{\partial_x y}{L} \right) - \partial_t \left(\frac{\partial_t y}{L} \right) \\
&= \frac{1}{L} \left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} \right) + \frac{1}{L^2} \left(\partial_t y\ \partial_t L - \partial_x y\ \partial_x L \right) \\
&= \frac{1}{L} \left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} \right) - \frac{1}{L^3} \left((\partial_t y)^2\, \frac{\partial^2 y}{\partial t^2} - 2\, \partial_x y\ \partial_t y\, \frac{\partial^2 y}{\partial x \partial t} + (\partial_x y)^2\, \frac{\partial^2 y}{\partial x^2} \right),
\end{aligned}$$

or

$$\begin{aligned}
0 &= L^2 \left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} \right) - \left((\partial_t y)^2\, \frac{\partial^2 y}{\partial t^2} - 2\, \partial_x y\ \partial_t y\, \frac{\partial^2 y}{\partial x \partial t} + (\partial_x y)^2\, \frac{\partial^2 y}{\partial x^2} \right) \\
&= \left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} \right) - \left((\partial_x y)^2\, \frac{\partial^2 y}{\partial t^2} - 2\, \partial_x y\ \partial_t y\, \frac{\partial^2 y}{\partial x \partial t} + (\partial_t y)^2\, \frac{\partial^2 y}{\partial x^2} \right).
\end{aligned}$$

Note that this is the wave equation for the simple elastic string with some additional cubic terms.

Substituting the an arbitrary right-running wave $y = f(x-t)$ into the above equation, we have

$$\begin{aligned}
0 &= \Big(f''(x-t) - (-1)^2 f''(x-t) \Big) - (-1)^2 \Big((f'(x-t))^2 f''(x-t) \\
&\quad - 2(f'(x-t))^2 f''(x-t) + (f'(x-t))^2 f''(x-t) \Big) \\
&= 0\,,
\end{aligned} \tag{1}$$

which means that an arbitrary right-running wave $y = f(x-t)$ is a solution to the equation of motion. Because Equation (1) is still satisfied under the transformation $-1 \to 1$, an arbitrary left-running wave $y = g(x+t)$ is also a solution to the equation of motion.

However, due to the additional cubic terms the linear combination $y = \alpha f(x-t) + \beta g(x+t)$ is not a solution. To show this explicitly, we substitute $y = \alpha f(x-t) +$

$\beta g(x+t)$ into the second parenthesis of the equation of motion, and we obtain

$$
\begin{aligned}
&(\partial_x y)^2 \,\frac{\partial^2 y}{\partial t^2} - 2\,\partial_x y\,\partial_t y\,\frac{\partial^2 y}{\partial x\partial t} + (\partial_t y)^2\,\frac{\partial^2 y}{\partial x^2} \\
&= (\alpha f' + \beta g')^2(\alpha f'' + \beta g'') - 2(\alpha f' + \beta g')(\alpha f' - \beta g')(\alpha f'' - \beta g'') \\
&+ (\alpha f' - \beta g')^2(\alpha f'' + \beta g'') \\
&= 4\alpha\beta\,\left(\alpha (f')^2 g'' + \beta (g')^2 f''\right) \\
&\neq 0\,.
\end{aligned}
$$

3. Because the stress-energy tensor $T^{\alpha\beta}(x)$ only has x dependence in the delta function, we have

$$
\begin{aligned}
\partial_\beta T^{\alpha\beta}(x) &= \int (L\,dp\,dq)\; v^{\alpha\gamma} {v_\gamma}^\beta \partial_\beta \delta^4(x - s(p,q)) \\
&= \int dp\,dq\; v^{\alpha\gamma} {a_\gamma}^\beta \partial_\beta \delta^4(x - s(p,q)) \\
&= \frac{1}{\sqrt{2}} \int dp\,dq\; v^{\alpha\gamma}\left(\partial_p s_\gamma\,\partial_q s^\beta - \partial_q s_\gamma\,\partial_p s^\beta\right)\partial_\beta \delta^4(x - s(p,q)) \\
&= \frac{1}{\sqrt{2}} \int dp\,dq\; v^{\alpha\gamma}\left(-\partial_p s_\gamma\,\partial_q \delta^4(x - s(p,q)) + \partial_q s_\gamma\,\partial_p \delta^4(x - s(p,q))\right).
\end{aligned}
$$

Here, we have applied the chain rule

$$
\partial_a \delta^4(x - s(p,q)) = -\partial_a s^\beta\,\partial_\beta \delta^4(x - s(p,q)).
$$

Applying integration by part, we have

$$
\begin{aligned}
&\int dp\,dq\; v^{\alpha\gamma}\left(\partial_q s_\gamma\,\partial_p \delta^4(x - s(p,q)) - \partial_p s_\gamma\,\partial_q \delta^4(x - s(p,q))\right) \\
&= \int dq\;\left(v^{\alpha\gamma}\,\partial_q s_\gamma\;\delta^4(x - s(p,q))\right)\Big|_{p_1}^{p_2} - \int dp\;\left(v^{\alpha\gamma}\,\partial_p s_\gamma\;\delta^4(x - s(p,q))\right)\Big|_{q_1}^{q_2} \\
&- \int dp\,dq\;\left[\partial_p\left(v^{\alpha\gamma}\,\partial_q s_\gamma\right) - \partial_q\left(v^{\alpha\gamma}\,\partial_p s_\gamma\right)\right]\delta^4(x - s(p,q)).
\end{aligned}
$$

By having x avoid the boundaries of the world surface, the first two terms vanish, and $\partial_\beta T^{\alpha\beta}(x)$ becomes

$$
\begin{aligned}
\partial_\beta T^{\alpha\beta}(x) &= -\frac{1}{\sqrt{2}} \int dp\,dq\;\left[\partial_p\left(v^{\alpha\gamma}\,\partial_q s_\gamma\right) - \partial_q\left(v^{\alpha\gamma}\,\partial_p s_\gamma\right)\right]\delta^4(x - s(p,q)) \\
&= -\frac{1}{\sqrt{2}} \int dp\,dq\;\left[(\partial_p v^{\alpha\gamma})\,(\partial_q s_\gamma) - (\partial_q v^{\alpha\gamma})\,(\partial_p s_\gamma)\right]\delta^4(x - s(p,q)) \\
&= -\frac{1}{\sqrt{2}} \int dp\,dq\;\left[\epsilon^{ab}(\partial_a v^{\alpha\gamma})(\partial_b s_\gamma)\right]\;\delta^4(x - s(p,q)) \\
&= 0.
\end{aligned}
$$