Assignment 12 solutions

1. Substituting $u(\theta) = (1 + \epsilon \cos\theta)/r_0$ into the energy

$$E = \frac{L_z^2}{2\mu}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) - Au\,,$$

we have

$$\begin{aligned} E &= \frac{L_z^2}{2\mu r_0^2}\left(\epsilon^2\sin^2\theta + (1+\epsilon\cos\theta)^2\right) - \frac{A}{r_0}\,(1+\epsilon\cos\theta) \\ &= \frac{L_z^2}{2\mu r_0^2}\left(\epsilon^2 + 2\epsilon\cos\theta + 1\right) - \frac{A}{r_0}\,(1+\epsilon\cos\theta) \\ &= \frac{A}{2r_0}\left(\epsilon^2 + 2\epsilon\cos\theta + 1\right) - \frac{A}{r_0}\,(1+\epsilon\cos\theta) \\ &= \frac{A}{2r_0}(\epsilon^2 - 1)\;. \end{aligned}$$

2. Equating the two expressions of A^2, we obtain

$$\begin{aligned} \pi^2 a^2 b^2 &= \frac{L_z^2}{4\mu^2}T^2 \\ &= \frac{G(M_1+M_2)}{4}\,r_0 T^2\;. \end{aligned} \tag{1}$$

Recall that

$$\begin{aligned} r_{min} &= a - \sqrt{a^2-b^2} = r_0/(1+\epsilon) \\ r_{max} &= a + \sqrt{a^2-b^2} = r_0/(1-\epsilon) \\ \epsilon &= \sqrt{a^2-b^2}/a\;. \end{aligned}$$

The product of r_{min} and r_{max}

$$\begin{aligned} r_{min}\cdot r_{max} &= b^2 \\ &= r_0^2/(1-\epsilon^2) \\ &= r_0^2\,a^2/b^2 \end{aligned}$$

gives us the relation $b^2 = r_0\,a$.

Substituting this back to Equation (1), we can readily get Kepler's 2^{nd} law:

$$T^2 = \frac{4\pi^2}{G(M_1+M_2)}\,a^3\;.$$

3. • **Method 1:**
The space probe launched with velocity parallel to the instantaneous velocity of the Earth and of a magnitude such that the aphelion of its elliptic orbit equals to r_M, so the point it launched is the perihelion of its orbit and we have $r_{min} = r_E$ and $r_{max} = r_M$. From the relations that

$$r_{min} = r_0/(1+\epsilon) = r_E$$
$$r_{max} = r_0/(1-\epsilon) = r_M = \frac{3}{2} r_E\,,$$

we can solve $\epsilon = \frac{1}{5}$.
From Kepler's 2^{nd} law and the fact that the mass of the sun is much larger than that of the Earth, Mars, or the probe, we have

$$\frac{a_p^3}{T_p^2} = \frac{r_E^3}{T_E^2} = \frac{r_M^3}{T_M^2}\,,$$

where a_p and T_p denote the semi-major axis length and the orbital period of the probe respectively. Substituting the relation that $r_M = \frac{3}{2} r_E$, the equation above becomes

$$\frac{(5r_E/4)^3}{T_p^2} = \frac{r_E^3}{T_E^2} = \frac{(3r_E/2)^3}{T_M^2}\,.$$

We can therefore solve T_p and T_M as

$$T_p = \frac{5\sqrt{5}}{8}\,T_E\,,\; T_M = \frac{3\sqrt{6}}{4}\,T_E\,.$$

Because the orbits of the Earth and Mars are circular, we have

$$v_M = \frac{2\pi r_M}{T_M} = \frac{\sqrt{6}}{3}\,\frac{2\pi r_E}{T_E} = \frac{\sqrt{6}}{3}\,v_E\,,$$

where v_E and v_M are the orbital speed of the Earth and Mars respectively.
By equating the dynamical and geometric formulas for the orbit area as what we have done in problem 2

$$\frac{L_z}{2\mu}\,T = \pi a b\,,$$

the orbital angular momentum can be expressed as

$$L_z = \frac{2\pi\mu a b}{T}. \tag{2}$$

Therefore, the orbital angular momentum of the probe with respect to the sun is given by

$$L_z = \frac{2\pi\mu a b}{T_p} = \frac{2\pi\mu\, a^2\sqrt{1-\epsilon^2}}{T_p} = \frac{\sqrt{30}}{5}\mu\, r_E v_E\,.$$

Because the angular momentum of the probe is conserved in this elliptic orbit, we have

$$L_z = \mu\, r_{min} v_{min} = \mu\, r_{max} v_{max}\ . \tag{3}$$

We can hence solve v_{min} and v_{max} as

$$v_{min} = \frac{\sqrt{30}}{5}\, v_E\ ,\ v_{max} = \frac{2\sqrt{30}}{15}\, v_E\ .$$

Since the probe's arrival of the orbit of Mars is timed so that its velocity is parallel to the instantaneous velocity of Mars, we obtain the relation that

$$v'_p = v_{max}\left(\frac{2v_M}{v_{max}} - 1\right) = \left(\frac{2\sqrt{6}}{3} - \frac{2\sqrt{30}}{15}\right) v_E\ ,$$

where v'_p is the magnitude of the velocity of the probe right after its encounter with Mars.

After the encounter, the probe has a new orbit and the point of encounter becomes the perihelion of this new orbit. We hence have $r'_{min} = r_M$ and $v'_{min} = v'_p$. The new orbital angular momentum of the probe with respect to the sun is given by

$$L'_z = \frac{2\pi\mu a' b'}{T'_p} = \frac{2\pi\mu\, a'^2\sqrt{1-\epsilon'^2}}{T'_p} = \mu\, r'_{min} v'_{min} = \mu\, \frac{b'^2}{a'(1+\epsilon')} v'_{min}.$$

Simplifying the equation above, we have

$$\frac{2\pi a'}{T'_p} = \sqrt{\frac{1-\epsilon'}{1+\epsilon'}}\, v'_{min}. \tag{4}$$

From Kepler's 2^{nd} law, we can relate the probe's new orbit with the orbit of Mars by

$$\frac{4\pi^2 a'^3}{T'^2_p} = \frac{4\pi^2 r_M^3}{T_M^2} = r_M v_M^2. \tag{5}$$

Combining Equation (4) and (5), we arrive at the relation

$$a'\frac{1-\epsilon'}{1+\epsilon'}\, v'^2_{min} = r_M v_M^2 = r'_{min} v_M^2 = a'(1-\epsilon') v_M^2\ ,$$

which gives us

$$v'^2_{min}/v_M^2 = 1 + \epsilon'.$$

Substituting the values of v'_{min} and v_M, we can solve ϵ' as

$$\epsilon' = \frac{19}{5} - \frac{8\sqrt{5}}{5}\ ,$$

and we hence obtain

$$r'_{max}/r_M = r'_{max}/r'_{min} = \frac{1+\epsilon'}{1-\epsilon'} = \frac{4+20\sqrt{5}}{31} \sim 1.57.$$

- **Method 2:**
The space probe launched with velocity parallel to the instantaneous velocity of the Earth and of a magnitude such that the aphelion of its elliptic orbit equals to r_M, so the point it launched is the perihelion of its orbit and we have $r_{min} = r_E$ and $r_{max} = r_M$. From the relations that

$$r_{min} = r_0/(1+\epsilon) = r_E$$
$$r_{max} = r_0/(1-\epsilon) = r_M = \frac{3}{2} r_E \,,$$

we can solve $\epsilon = \frac{1}{5}$.
From the result of Problem 1, the total energy of an object moving in an elliptic orbit is given by

$$E = \frac{A}{2r_0}(\epsilon^2 - 1) = -\frac{A}{2r_{min}}(1-\epsilon) \ .$$

At the perihelion, we hence have

$$\frac{1}{2}\mu v_{min}^2 - \frac{A}{r_{min}} = -\frac{A}{2r_{min}}(1-\epsilon) \,,$$

and we can solve

$$v_{min} \simeq \sqrt{\frac{Gm_S}{r_{min}}\,(1+\epsilon)} \,, \tag{6}$$

where m_S denotes the mass of the sun and '$\simeq$' is owing to the fact that the mass of the sun is much larger than that of the Earth, Mars, or the probe.
From Equation (6), we can relate an elliptic orbit and a circular orbit ($\epsilon = 0$) which have the same perihelion by

$$(v_{min}/v_0)^2 = 1 + \epsilon \ . \tag{7}$$

Therefore, the launching speed of the probe is given by

$$v_{min} = \frac{\sqrt{30}}{5}\, v_E \ .$$

By the orbital angular momentum conservation

$$L_z = \mu\, r_{min} v_{min} = \mu\, r_{max} v_{max} \,,$$

we can solve

$$v_{max} = \frac{2\sqrt{30}}{15}\, v_E$$

as the speed of the probe at the aphelion.

From Kepler's 2^{nd} law, the orbits of the earth and Mars are related by

$$\frac{r_M^3}{T_M^2} = \frac{r_E^3}{T_E^2} ,$$

from which together with $r_M = \frac{3}{2} r_E$ we can solve

$$v_M = \frac{\sqrt{6}}{3} v_E .$$

Since the probe's arrival of the orbit of Mars is timed so that its velocity is parallel to the instantaneous velocity of Mars, we obtain the relation that

$$v_p' = v_{max} \left(\frac{2v_M}{v_{max}} - 1 \right) = \left(\frac{2\sqrt{6}}{3} - \frac{2\sqrt{30}}{15} \right) v_E ,$$

where v_p' is the magnitude of the velocity of the probe right after its encounter with Mars.

After the encounter, the probe has a new orbit and the point of encounter becomes the perihelion of this new orbit. We hence have $r'_{min} = r_M$ and $v'_{min} = v'_p$. Using Equation (7), we can relate v'_{min} and v_M by

$$(v'_{min}/v_M)^2 = 1 + \epsilon' ,$$

from which we can solve ϵ' as

$$\epsilon' = \frac{19}{5} - \frac{8\sqrt{5}}{5} ,$$

and we hence obtain

$$r'_{max}/r_M = r'_{max}/r'_{min} = \frac{1 + \epsilon'}{1 - \epsilon'} = \frac{4 + 20\sqrt{5}}{31} \sim 1.57.$$