Physics 6561 Fall 2017
Problem Set 11 Solutions

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Non-adiabatic loss from a magnetic trap

Using the results of the last problem of assignment 10 and Fermi's Golden Rule, complete the calculation begun in lecture, of the differential rate

$$\frac{d\Gamma}{d\Omega} = Cf(\theta,\phi)e^{-2\omega_p/\omega_0} \tag{1.1}$$

at which the trapped spin-1/2 particle exits the trap along polar angles (θ, ϕ) with respect to the direction of the magnetic field at the center of the trap. Express C only in terms of fundamental constants, the particle mass m and magnetic moment μ, and the oscillator and precession frequencies (ω_0 and ω_p) instead of the trap parameters (isotropic case).
Interpret the angular function $f(\theta, \phi)$ in terms of a conservation law.

Due to Chris Wilson, edited by Amir

Our initial state, which is the ground state is given by

$$|\uparrow, n=0\rangle = C|\uparrow\rangle e^{-\frac{-m\omega_0}{2\hbar}|\vec{x}|^2} \tag{1.2}$$

where one can find normalization constant C by normalization of wavefunction

$$\langle\uparrow, n=0|\uparrow, n=0\rangle = 1 \quad |C|^2\langle\uparrow|\uparrow\rangle\int e^{-\frac{-m\omega_0}{\hbar}|\vec{x}|^2}d^3x = 1$$
$$|C|^2\left(\frac{\pi\hbar}{m\omega_0}\right)^{3/2} = 1 \qquad C = \left(\frac{m\omega_0}{\pi\hbar}\right)^{3/4} \tag{1.3}$$

While final states are plane wave states:

$$|\downarrow, \vec{k}\rangle = \frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{x}}|\downarrow\rangle \tag{1.4}$$

Where V is the volume of space.
From the previous homework, we know interaction hamiltonian has the form

$$\hat{H}_{int} = -\frac{\hbar^2}{2m}\left(\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{A} + 2\mathbf{A} \cdot \nabla\right) \tag{1.5}$$

To lowest order in $1/B$, only the $2\mathbf{A} \cdot \nabla$ term has off-diagonal terms and hence only this term contributes to transition amplitude. More explicitly, we have

$$A_x = \frac{\beta}{2B}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad A_y = \frac{\beta}{2B}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \qquad A_z = 0 \tag{1.6}$$

As it's derived in the lecture, Fermi's golden rule determines the transition amplitude from up state to down state as following

$$\frac{d\Gamma}{d\Omega} = \frac{2\pi}{\hbar}\frac{Vk^2}{(2\pi)^3}\left|\frac{dk}{dE_F}\right|\left|\langle\uparrow, n=0|\hat{H_{int}}|\downarrow, \vec{k}\rangle\right|^2 \tag{1.7}$$

where E_F is the Fermi energy, and $\vec{k}$ denotes momentum of particles

$$E_F(k) = \frac{\hbar^2 k^2}{2m} \quad \Rightarrow \left|\frac{dE_F}{dk}\right| = \frac{\hbar^2 k}{m} \tag{1.8}$$

Since Fermi's golden rule equates initial energies to final energies, energy gap should be equal to energy of particles (this assumes $\omega_p \gg \omega_0$, so we can ignore zero energy of initial state). Thus,

$$\frac{\hbar^2 k^2}{2m} = \hbar\omega_p \qquad \Rightarrow \quad k = \left(\frac{2m\omega_p}{\hbar}\right)^{1/2} \tag{1.9}$$

Now we only need to compute $\left|\langle\uparrow, n=0|\hat{H}_{int}|\downarrow, \vec{k}\rangle\right|$:

$$\langle\uparrow, n=0|\hat{H_{int}}|\downarrow,\vec{k}\rangle = \frac{\hbar^2 C^\star}{m\sqrt{V}}\langle\uparrow, n=0|\hat{A}_x\partial_x + \hat{A}_y\partial_y + \hat{A}_z\partial_z|\downarrow,\vec{k}\rangle$$
$$= \int d^3x e^{-\frac{m\omega_0}{2\hbar}\vec{x}^2} e^{i\vec{k}\cdot\vec{x}} \frac{\hbar^2 C^\star}{m\sqrt{V}}\langle\uparrow|ik_x\hat{A}_x + ik_y\hat{A}_y|\downarrow\rangle$$
$$\frac{\hbar^2 C^\star}{m\sqrt{V}}\int d^3x\, e^{-\frac{m\omega_0}{2\hbar}\vec{x}^2 + i\vec{k}\cdot\vec{x}}\left(\frac{\beta}{2B}(-ik_x + k_y)\right)$$
$$= \frac{\hbar^2\beta C^\star}{2mB\sqrt{V}}(-ik_x + k_y)\int d^3x\, e^{-\frac{m\omega_0}{2\hbar}(\vec{x}+i\frac{\hbar}{m\omega_0}\vec{k})^2 - \frac{\hbar^2k^2}{2m\omega_0}}$$
$$\frac{\hbar^2\beta C^\star}{2mB\sqrt{V}}(-ik_x + k_y)\sqrt{\frac{2\pi\hbar}{m\omega_0}}e^{-\frac{\hbar^2k^2}{2m\omega_0}} = \frac{\hbar^2\beta C^\star}{2mB\sqrt{V}}(-ik_x + k_y)\left(\frac{2\pi\hbar}{m\omega_0}\right)^{3/2} e^{-\omega_p/\omega_0} \tag{1.10}$$

where $\star$ means complex conjugate.
Using relation between frequencies and magnetic field

$$\omega_0 = \sqrt{\frac{\gamma\mu}{m}} \qquad \text{in isotropic case} \qquad = \sqrt{\frac{2\beta^2\mu}{3mB}}$$
$$B = \frac{2\mu\beta^2}{3m} \qquad \frac{\beta^2}{B^2} = \frac{3\gamma}{2B} = \frac{3m\omega_0^2}{2\mu B} = \frac{3m\omega_0^2}{\hbar\omega_p} \tag{1.11}$$

we can write the final expression for decay rate

$$\frac{d\Gamma}{d\Omega} = \frac{2\pi}{\hbar}\frac{Vk^2}{(2\pi)^3}\frac{m}{\hbar^2 k}\frac{\hbar^4\beta^2|C|^2}{4m^2B^2V}|-ik_x + k_y|^2\left(\frac{2\pi\hbar}{m\omega_0}\right)^3 e^{-2\omega_p/\omega_0}$$
$$= \left(\frac{\pi\hbar}{m\omega_0}\right)^{3/2}\frac{\hbar\beta^2}{2\pi^2 mB^2}k|-ik_x + k_y|^2 e^{-2\omega_p/\omega_0}$$
$$= \left(\frac{\pi\hbar}{m\omega_0}\right)^{3/2}\frac{3\omega_0^2}{2\pi^2\omega_p}k^3\sin^2(\theta)e^{-2\omega_p/\omega_0}$$
$$= \left(\frac{\pi\hbar}{m\omega_0}\right)^{3/2}\left(\frac{2m\omega_p}{\hbar}\right)^{3/2}\frac{3\omega_0^2}{2\pi^2\omega_p}\sin^2(\theta)e^{-2\omega_p/\omega_0}$$
$$= 3\sqrt{\frac{2\omega_0\omega_p}{\pi}}\sin^2(\theta)e^{-2\omega_p/\omega_0} \tag{1.12}$$

One can check explicitly that total angular momentum in the z-direction, $\hat{J}_z = \hat{S}_z + \hat{L}_z$, in which $\hat{S}$ and $\hat{L}$ are spin and orbital angular momentum respectively, commutes with interaction Hamiltonian and that means interaction will preserve conservation of total angular momentum in the z-direction.

Initially, only spin part of angular momentum is non-zero since orbital angular momentum vanishes for isotropic Gaussian wave-function. When transition happens, the spin of particles changes from $\hat{S}_z = +1/2$ to $\hat{S}_z = -\frac{1}{2}$, therefore the orbital angular momentum's value should be equal to +1. Interestingly, the function appeared above, $-ik_x + k_y$ is in fact proportional to $l = 1$ spherical harmonics:

$$-ik_x + k_y = -ik\sin(\theta)e^{i\phi} = ik\sqrt{\frac{8\pi}{3}}Y_{1,1}(\theta,\phi) \tag{1.13}$$

Consistent with conservation of total angular momentum.

Gauge transformation of the magnetic field in Yang-Mills theory

Apply the Yang-Mills generalization of gauge transformations,

$$\begin{aligned} A'_x &= UA_xU^\dagger - iU(\partial_x U^\dagger) \\ A'_y &= UA_yU^\dagger - iU(\partial_y U^\dagger) \end{aligned} \tag{1.14}$$

to the Yang-Mills generalization of the magnetic field

$$B_z = \partial_x A_y - \partial_y A_x + i[A_x, A_y] \tag{1.15}$$

Treat A_x, A_y, B_z and U as non-commuting matrices, as in the analysis of the magnetic trap. Using only the group property $UU^\dagger = \mathbb{1}$ (and its derivatives) show that

$$B'_z = UB_zU^\dagger \tag{1.16}$$

Solution due to Daniel Longenecker

We need to show $U^\dagger B'_z U = B_z$.

$$
\begin{aligned}
&U^\dagger\partial_x(UA_yU^\dagger)U + U^\dagger\partial_x(-iU\partial_yU^\dagger)U - U^\dagger\partial_y(UA_xU^\dagger)U - U^\dagger\partial_y(-iU\partial_xU^\dagger)U \\
&+ iU^\dagger\Big((UA_xU^\dagger - iU(\partial_xU^\dagger))(UA_yU^\dagger - iU(\partial_yU^\dagger)) \\
&-(UA_yU^\dagger - iU(\partial_yU^\dagger))(UA_xU^\dagger - iU(\partial_xU^\dagger))\Big)U \\
&= U^\dagger(\partial_xU)A_y + \partial_xA_y + A_y(\partial_xU^\dagger)U - iU^\dagger\partial_xU(\partial_yU^\dagger)U \\
&- i(\partial_x\partial_yU^\dagger)U - U^\dagger\partial_yUA_x \\
&- \partial_yA_x - A_x\partial_yU^\dagger U + iU^\dagger\partial_yU\partial_x + i\partial_y\partial_xU^\dagger U \\
&+ i[A_x, A_y] + A_x\partial_yU^\dagger U + \partial_xU^\dagger UA_y - i\partial_xU^\dagger U\partial_yU^\dagger U + \\
&i\partial_yU^\dagger U\partial_xU^\dagger U - A_y\partial_xU^\dagger U - \partial_yU^\dagger UA_x \\
&\partial_xA_y - \partial_yA_x + i[A_x, A_y] + \partial_x(U^\dagger U)A_y - \partial_y(U^\dagger U)A_x \\
&- i\partial_x(U^\dagger U)\partial_yU^\dagger U + i\partial_y(U^\dagger U)\partial_xU^\dagger U \\
&= \partial_xA_y - \partial_yA_x + i[A_x, A_y] = B_z
\end{aligned}
\tag{1.17}
$$

Where I used $\partial_{x_i}(U^\dagger U) = \partial_{x_i}(UU^\dagger) = 0$.