Assignment 11 solutions

1. (a) The equilibrium occurs when the first derivative of $U(r)$ vanishes.

$$0 = U'(r) = -\frac{2B}{r^3} + \frac{A}{r^2} + \frac{3C}{r^4}.$$  

Regrouping the equation above, we have

$$r = \frac{2B}{A} - \frac{3C}{Ar} = r_0 - \frac{3C}{Ar}.$$  \hspace{1cm} (1)

Because $C$ is only a tiny correction to the potential, we can obtain the solution to $r$ in power series of $C$ with perturbation theory. For the zeroth-order solution, we simply neglect the correction that involves $C$, and we end up with $r = r_0$. For the first-order solution, we substitute the zeroth-order solution $r = r_0$ into the right hand side of Equation (1), and we obtain

$$r = r_1 = r_0 - \frac{3C}{2B}.$$  

Note that we can refine the solution to $r$ to an arbitrarily high order in $C$ by iteratively performing the procedure above.

(b) The second order derivative of $U(r)$ is given by

$$U''(r) = \frac{6B}{r^4} - \frac{2A}{r^3} - \frac{12C}{r^5}.$$  

To simplify the notation, we define

$$r_1 = r_0 - \frac{3C}{2B} = r_0 (1 - \frac{3A}{4B^2} C) \equiv r_0 (1 + c_1 C).$$

At the equilibrium $r = r_1$, we have

$$K = U''(r_1) = \frac{6B}{r_0^4 (1 + c_1 C)^4} - \frac{2A}{r_0^3 (1 + c_1 C)^3} - \frac{12C}{r_0^5 (1 + c_1 C)^5}$$

$$= \frac{6B}{r_0^4} (1 - 4c_1 C) - \frac{2A}{r_0^3} (1 - 3c_1 C) - \frac{12C}{r_0^5} + O(C^2)$$

$$= \frac{A}{r_0^3} + \frac{6C}{r_0^5} + O(C^2).$$

Hence, we find the value of $\alpha$ to be 6.
(c) The angular velocity for orbit completion \( \omega_\theta \) is given by

\[
\omega_\theta = \frac{L_z}{\mu r^2} = \frac{\sqrt{2 \mu B}}{\mu r_0^2 (1 + c_1 C)^2} + O(C^2)
\]

\[
= \frac{\sqrt{2 \mu B}}{\mu r_0^2} (1 - 2c_1 C) + O(C^2)
\]

\[
= \sqrt{\frac{2B}{\mu}} \frac{1}{r_0^2} (1 + \frac{3A}{2B^2} C) + O(C^2).
\]

On the other hand, the angular velocity for radial oscillation \( \omega_r \) is given by

\[
\omega_r = \sqrt{\frac{K}{\mu}}
\]

\[
= \sqrt{\frac{1}{\mu}} \left( A \frac{6}{r_0^3} + \frac{6C}{r_0^5} + O(C^2) \right)^{\frac{1}{2}}
\]

\[
= \sqrt{\frac{A}{\mu r_0^3}} \left( 1 + \frac{3C}{Ar_0^2} \right) + O(C^2)
\]

\[
= \sqrt{\frac{2B}{\mu}} \frac{1}{r_0^2} \left( 1 + \frac{3A}{4B^2} C \right) + O(C^2).
\]

Therefore the ratio \( \omega_r / \omega_\theta \) can be expressed as

\[
\frac{\omega_r}{\omega_\theta} = (1 + \frac{3A}{4B^2} C)(1 + \frac{3A}{2B^2} C)^{-1} + O(C^2)
\]

\[
= (1 + \frac{3A}{4B^2} C)(1 - \frac{3A}{2B^2} C) + O(C^2)
\]

\[
= 1 - \frac{3A}{4B^2} C + O(C^2)
\]

\[
\sim 1 - \frac{3}{2Br_0} C.
\]

Therefore, the value of \( \beta \) is \(-3/2\).

(d) Because \( \omega_r \) is smaller than \( \omega_\theta \), the orbit sweeps through an angle larger than \( 2\pi \) over one course of radial oscillation. The excess angle \( \delta \theta \) can be obtained by solving the equation

\[
2\pi = (2\pi + \delta \theta) \left( 1 - \frac{3}{2Br_0} C \right).
\]

Therefore,

\[
\delta \theta \sim 2\pi \left( 1 + \frac{3}{2Br_0} C \right) - 2\pi = \frac{3\pi}{Br_0} C.
\]
2. From the numerical values of $T_I$, $T_E$ and $T_G$, we can solve the values of $\omega_0$ and $\omega_p$ as

$$\omega_0 = 0.89111477 \, \text{Day}^{-1}$$
$$\omega_p = -0.01290681 \, \text{Day}^{-1},$$

whose accuracy is up to hundred millionths!

After one period $T_0 = 2\pi/\omega_0$, the moons sweep through angles

$$\theta_I = 8\pi + \omega_p T_0$$
$$\theta_E = 4\pi + \omega_p T_0$$
$$\theta_G = 2\pi + \omega_p T_0$$

The moons tend to line up with one another to minimize their potential energy, which explains the nearly integral ratio 1:2:4 in their periods. However, the small deviation $\omega_p$ shows that the orbits are precessing at the same time. If the gravitational quadrupole potential $C/r^3$ of Jupiter is the cause, it will influence the innermost moon most. From Kepler’s 1-2-3 law, the orbit radii are proportional to $\omega_0^{-2/3}$ if the moons orbit around the same massive planet. Therefore, we know that the innermost moon must be Io. Because the moons tend to line up to minimize their potential energy, the interactions between the moons would also perturb the orbits of Europa and Ganymede to offset their angular frequencies by the same amount $\omega_p$. This phenomenon is called “orbital resonance”.

From Equation(2) in Problem 1(c), $\omega_p$ must be positive if the quadrupole potential of Jupiter causes the orbital precession. However, $\omega_p$ is negative in this question, and this indicates that the orbital precession is caused by something else, in this case, the interactions between the moons.

3. Neglecting the constant center-of-mass kinetic energy, the Lagrangian can be written as

$$L = \frac{1}{2} \mu \ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} - Ar^2$$
$$= \frac{1}{2} \mu (\dot{x}^2 + \dot{y}^2) - A(x^2 + y^2).$$

The Euler-Lagrange equations give us

$$\mu \ddot{x} = -2Ax$$
$$\mu \ddot{y} = -2Ay.$$ 

We can readily solve $x(t)$ and $y(t)$ as

$$x(t) = x_0 \cos(\omega_0 t + \phi_x)$$
$$y(t) = y_0 \cos(\omega_0 t + \phi_y),$$
where $\omega_\theta = \sqrt{2A/\mu}$ is the angular frequency for orbital completion, and $x_0, y_0, \phi_x$ and $\phi_y$ are constants determined by the initial conditions. This trajectory $(x(t), y(t))$ is an ellipse with the center of mass located at the center, so the orbit is closed. Note that the radial oscillation sweeps through two periods as the orbit completes one period, or $\omega_r/\omega_\theta = 2$. 