

Assignment 10 solutions

1. Start with the formula for the volume of the entire accessible phase space in D dimensions

$$\begin{aligned} \text{vol}(\Delta E) &= \int_{E_0-\Delta E/2}^{E_0+\Delta E/2} dE \int d^D \mathbf{x} \int d^D \mathbf{p} \delta(H(\mathbf{x}, \mathbf{p}) - E) \\ &= \Omega_D \int_{E_0-\Delta E/2}^{E_0+\Delta E/2} dE \int d^D \mathbf{x} \int dp p^{D-1} \delta(H(\mathbf{x}, \mathbf{p}) - E) . \end{aligned}$$

Under the change of variable

$$E' = H(\mathbf{x}, \mathbf{p}) - E = \frac{p^2}{2M} + V(\mathbf{x}) - E ,$$

we have

$$\begin{aligned} \int dp p^{D-1} \delta\left(\frac{p^2}{2M} + V(\mathbf{x}) - E\right) &= \int dE' M p^{D-2} \delta(E') \\ &\propto \int dE' |E' - (V(\mathbf{x}) - E)|^{(D-2)/2} \delta(E') \\ &\propto |E - V(\mathbf{x})|^{(D-2)/2} . \end{aligned}$$

Hence, we obtain

$$\text{vol}(\Delta E) \propto \int_{E_0-\Delta E/2}^{E_0+\Delta E/2} dE \int d^D \mathbf{x} |E - V(\mathbf{x})|^{(D-2)/2} .$$

- (a) $D = 2$:

$$\text{vol}(\Delta E) \propto \int_{E_0-\Delta E/2}^{E_0+\Delta E/2} dE \int d^2 \mathbf{x} \propto \int d^2 \mathbf{x} .$$

Because each volume element has the same constant density of states $\rho(\mathbf{x}) = \rho_0$, each position \mathbf{x} should be visited at the same rate under the ergodic hypothesis. This is consistent with the numerical result of the simple Yang-Mills oscillator we obtained in Assignment 9.

- (b) $D \neq 2$:

$$\begin{aligned} \text{vol}(\Delta E) &\propto \int_{E_0-\Delta E/2}^{E_0+\Delta E/2} dE \int d^D \mathbf{x} |E - V(\mathbf{x})|^{(D-2)/2} \\ &\propto \int d^D \mathbf{x} \left(\left| E_0 - V(\mathbf{x}) + \frac{\Delta E}{2} \right|^{D/2} - \left| E_0 - V(\mathbf{x}) - \frac{\Delta E}{2} \right|^{D/2} \right) . \end{aligned}$$

We can then expand the integrand in terms of ΔE :

$$\begin{aligned} & \left| E_0 - V(\mathbf{x}) + \frac{\Delta E}{2} \right|^{D/2} - \left| E_0 - V(\mathbf{x}) - \frac{\Delta E}{2} \right|^{D/2} \\ &= |E_0 - V(\mathbf{x})|^{D/2} \left[\left| 1 + \frac{\Delta E}{2(E_0 - V(\mathbf{x}))} \right|^{D/2} - \left| 1 - \frac{\Delta E}{2(E_0 - V(\mathbf{x}))} \right|^{D/2} \right] \\ &= |E_0 - V(\mathbf{x})|^{D/2} \left(D \frac{\Delta E}{2|E_0 - V(\mathbf{x})|} + O((\Delta E)^2) \right). \end{aligned}$$

i. $D = 1$:

$$\text{vol}(\Delta E) \propto \int dx |E_0 - V(x)|^{-1/2}.$$

The positions with higher potential energy have higher density of states

$$\rho(x) = |E_0 - V(x)|^{-1/2},$$

and are therefore visited more frequently. This is precisely the behavior we expect of the 1D simple harmonic oscillator.

ii. $D > 2$:

$$\text{vol}(\Delta E) \propto \int d^D \mathbf{x} |E_0 - V(\mathbf{x})|^{(D-2)/2}.$$

The positions with lower potential energy have higher density of states

$$\rho(\mathbf{x}) = |E_0 - V(\mathbf{x})|^{(D-2)/2},$$

so the particle spends more time in regions of high kinetic energy.

2. Because $F(\theta, I', t)$ is a generating function of the type F_3 , we have

$$I = \frac{\partial F}{\partial \theta} = \left(1 - \frac{\epsilon}{\omega(\epsilon t)} h(\theta, \epsilon t) \right) I' \quad (1)$$

$$\theta' = \frac{\partial F}{\partial I'} = \theta - \frac{\epsilon}{\omega(\epsilon t)} \int_0^\theta h(\tilde{\theta}, \epsilon t) d\tilde{\theta}. \quad (2)$$

The transformed Hamiltonian $H'(\theta', I', t)$ is hence given by

$$H'(\theta', I', t) = H(\theta, I, t) + \frac{\partial F(\theta, I', t)}{\partial t},$$

and we will need to use Equation (1) and (2) to simplify the expressions of $H(\theta, I, t)$ and $\partial F / \partial t$.

Since Equation (2) tells us that θ' is given by θ with some correction with order $O(\epsilon)$, we can approximate θ by $\theta = \theta' + O(\epsilon)$ according to the perturbation theory. As a result,

$$\begin{aligned} H(\theta, I, t) &= (\omega(\epsilon t) + \epsilon h(\theta, \epsilon t)) I \\ &= (\omega(\epsilon t) + \epsilon h(\theta, \epsilon t)) \left(1 - \frac{\epsilon}{\omega(\epsilon t)} h(\theta, \epsilon t) \right) I' \\ &= \left(\omega(\epsilon t) - \epsilon^2 \frac{[h(\theta, \epsilon t)]^2}{\omega(\epsilon t)} \right) I' \\ &= \left(\omega(\epsilon t) - \epsilon^2 \frac{[h(\theta', \epsilon t)]^2}{\omega(\epsilon t)} + O(\epsilon^3) \right) I' \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{\partial F(\theta, I', t)}{\partial t} &= \left(\epsilon^2 \frac{\omega'(\epsilon t)}{[\omega(\epsilon t)]^2} \int_0^\theta h(\tilde{\theta}, \epsilon t) d\tilde{\theta} - \frac{\epsilon^2}{\omega(\epsilon t)} \int_0^\theta \frac{\partial h(\theta, s)}{\partial s} \Big|_{(\tilde{\theta}, \epsilon t)} d\tilde{\theta} \right) I' \\ &= \epsilon^2 \left(\frac{\omega'(\epsilon t)}{[\omega(\epsilon t)]^2} \int_0^{\theta'} h(\tilde{\theta}, \epsilon t) d\tilde{\theta} - \frac{1}{\omega(\epsilon t)} \int_0^{\theta'} \frac{\partial h(\theta, s)}{\partial s} \Big|_{(\tilde{\theta}, \epsilon t)} d\tilde{\theta} + O(\epsilon) \right) I'. \end{aligned}$$

Therefore, we can rewrite the transformed Hamiltonian $H'(\theta', I', t)$ as

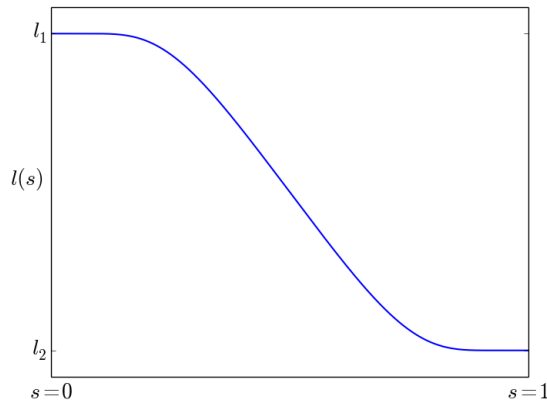
$$H'(\theta', I', t) = \left(\omega(\epsilon t) + \epsilon^2 h'(\theta', \epsilon t) + O(\epsilon^3) \right) I',$$

where

$$h'(\theta', \epsilon t) = -\frac{[h(\theta', \epsilon t)]^2}{\omega(\epsilon t)} + \frac{\omega'(\epsilon t)}{[\omega(\epsilon t)]^2} \int_0^{\theta'} h(\tilde{\theta}, \epsilon t) d\tilde{\theta} - \frac{1}{\omega(\epsilon t)} \int_0^{\theta'} \frac{\partial h(\theta, s)}{\partial s} \Big|_{(\tilde{\theta}, \epsilon t)} d\tilde{\theta}.$$

Notice that by applying such transformations iteratively, one can suppress the time dependence from the angular variable to arbitrary order in ϵ .

3.



The function $h(\theta, s)$ is given by

$$\begin{aligned} h(\theta, s) &= \frac{3}{2} \frac{1}{l} \frac{dl}{ds} \sin \theta \cos \theta \\ &= \frac{3}{2} \sin \theta \cos \theta \frac{d \log l(s)}{ds}. \end{aligned}$$

Therefore, showing that $h(\theta, s)$ and $\partial^n h / \partial s^n$ vanish at $s \rightarrow 0$ and $s \rightarrow 1$ is the same as showing that all the derivatives of $\log l(s)$ vanish at $s \rightarrow 0$ and $s \rightarrow 1$.

Define $x \equiv (s - 1/2)/(s(1 - s))$. As $s \rightarrow 0$, $x \rightarrow -\infty$. We can hence expand $f(s)$ in terms of e^{2x} :

$$\begin{aligned} f(s) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \frac{e^{2x} - 1}{e^{2x} + 1} \\ &= (e^{2x} - 1)(1 - e^{2x} + O(e^{4x})) \\ &= -1 + 2e^{2x} + O(e^{4x}). \end{aligned}$$

And $\log l(s)$ can be expressed as

$$\begin{aligned} \log l(s) &= \log \left[\frac{1}{2}(l_1 + l_2) + \frac{f(s)}{2}(l_2 - l_1) \right] \\ &= \log [l_1 + (l_2 - l_1)e^{2x} + O(e^{4x})] \\ &= \log l_1 + \log \left[1 + \frac{l_2 - l_1}{l_1} e^{2x} + O(e^{4x}) \right] \\ &= \log l_1 + \frac{l_2 - l_1}{l_1} e^{2x} + O(e^{4x}). \end{aligned}$$

Recall that $h(\theta, s)$ and $\partial^n h / \partial s^n$ are proportional to the derivatives of $\log l(s)$, and the leading order terms are all products of e^{2x} and the derivatives of $x(s)$. Because the derivatives of $x(s)$ only grow as a polynomial, the exponential decay e^{2x} always dominates no matter how large the power of the derivatives is. Therefore, $h(\theta, s)$ and $\partial^n h / \partial s^n$ all vanish at $s \rightarrow 0$.

As $s \rightarrow 1$, $x \rightarrow \infty$. We can expand $f(s)$ in terms of e^{-2x} :

$$\begin{aligned} f(s) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \frac{1 - e^{-2x}}{1 + e^{-2x}} \\ &= (1 - e^{-2x})(1 - e^{-2x} + O(e^{-4x})) \\ &= 1 - 2e^{-2x} + O(e^{-4x}). \end{aligned}$$

And $\log l(s)$ is given by

$$\begin{aligned}
 \log l(s) &= \log \left[\frac{1}{2}(l_1 + l_2) + \frac{f(s)}{2}(l_2 - l_1) \right] \\
 &= \log [l_2 - (l_2 - l_1)e^{-2x} + O(e^{-4x})] \\
 &= \log l_2 + \log \left[1 - \frac{l_2 - l_1}{l_2} e^{-2x} + O(e^{-4x}) \right] \\
 &= \log l_2 - \frac{l_2 - l_1}{l_2} e^{-2x} + O(e^{-4x}).
 \end{aligned}$$

As the same reason above, $h(\theta, s)$ and $\partial^n h / \partial s^n$ also vanish at $s \rightarrow 1$.