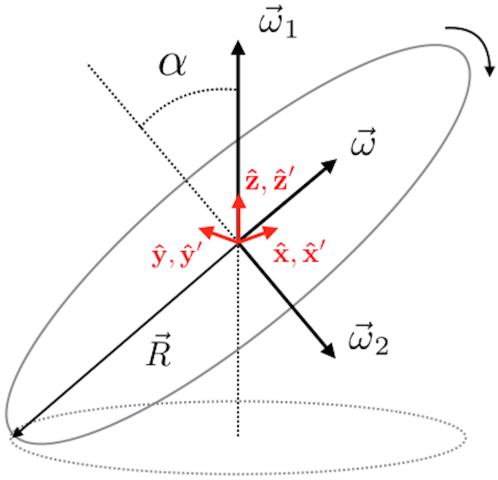
Assignment 1 solutions

1.



Consider the space frame (S) and the body frame (S') that coincide at time $t = 0$, with S' rotating at angular velocity $\vec{\omega}_1$ about S. The ring itself spins about its own axis at angular velocity $\vec{\omega}_2$. From the additivity of angular velocity, the angular velocity of points on the ring relative to S is

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2.$$

The ring rolls without slipping on the table, so the point of contact with the table has zero velocity in S:

$$\vec{v} = \vec{\omega} \times \vec{R} = \vec{0}.$$

Therefore, $\vec{\omega}$ should be parallel to the position vector $\vec{R}$ of the point of contact, and we have

$$\omega = \omega_1 \sin\alpha = \frac{2\pi}{T} \sin\alpha$$

Assume that $\vec{\omega}$ lies on xz plane at $t = 0$,

$$\begin{aligned}\vec{\omega}(t) &= \omega \left[\cos\alpha \cos(\omega_1 t)\ \hat{x} + \cos\alpha \sin(\omega_1 t)\ \hat{y} + \sin\alpha\ \hat{z}\right] \\ &= \frac{2\pi}{T} \sin\alpha \left[\cos\alpha \cos(\frac{2\pi}{T} t)\ \hat{x} + \cos\alpha \sin(\frac{2\pi}{T} t)\ \hat{y} + \sin\alpha\ \hat{z}\right].\end{aligned}$$

2. Relative to the ground, the velocity of the point of contact is

$$\vec{v} + \vec{\omega} \times (-R\,\hat{n}) = \vec{0},$$

or

$$R\,\vec{\omega} \times (\hat{n}) = \vec{v}. \tag{1}$$

In general, the three components of $\vec{\omega}$ and the three component of $\vec{v}$ of a rigid body can be specified independently. For the ball with the rolling-without-slipping constraint, Eq. (1) determines $\vec{v}$ completely from an arbitrary $\vec{\omega}$. Therefore we have only three degrees of freedom, as specified by $\vec{\omega}$.

3. The finite-difference integration of the equation $\dot{U} = AU$ is given by

$$U(t + \Delta t) = (\mathbb{1} + \Delta t A(t))\,U(t),$$

and $U(T)$ is approximated by

$$U(T) = \prod_{i=0}^{N-1} (\mathbb{1} + \Delta t A(i\Delta t))\,U(0) = \prod_{i=0}^{N-1} (\mathbb{1} + \Delta t A(i\Delta t)),$$

with $N \equiv T/\Delta t$.

In the gyrating ring problem,

$$A(t) = \begin{bmatrix} 0 & -\omega \sin\alpha & (\omega\cos\alpha)\sin\Omega t \\ \omega\sin\alpha & 0 & -(\omega\cos\alpha)\cos\Omega t \\ -(\omega\cos\alpha)\sin\Omega t & (\omega\cos\alpha)\cos\Omega t & 0 \end{bmatrix},$$

with $\omega = \Omega \sin\alpha$.

Using Python as an example, we can compute $U(T)$ by

```
import numpy as np
U = np.eye(3)
alpha = np.pi/4
Omega = 1.
w = Omega*np.sin(alpha)

def A(t):
    wx = w*np.cos(alpha)*np.cos(Omega*t)
    wy = w*np.cos(alpha)*np.sin(Omega*t)
    wz = w*np.sin(alpha)
    return np.array([[0., -wz, wy], \
                     [wz, 0., -wx], \
                     [-wy, wx, 0.]])
```

```
T = 2*np.pi/Omega
N = 10000
time_series = np.linspace(0, T, N)
dt = time_series[1] - time_series[0]
for i in xrange(N-1):
    t = time_series[i]
    U = U + dt*np.dot(A(t), U)

print U
print np.dot(U, U.T)
```

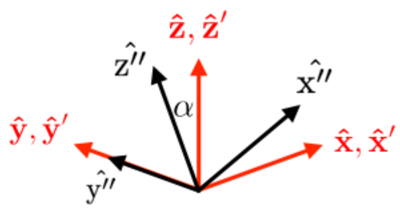
The program gives us

$$U(T) = \begin{bmatrix} 0.367 & -0.682 & -0.633 \\ 0.682 & -0.266 & 0.682 \\ -0.633 & -0.682 & 0.367 \end{bmatrix} \tag{2}$$

and

$$UU^T = \begin{bmatrix} 1.001 & 0.000 & -0.001 \\ 0.000 & 1.001 & 0.000 \\ -0.001 & 0.000 & 1.001 \end{bmatrix},$$

indicating that $U(T)$ is nearly orthogonal.



We can check our answer by calculating $U(T)$ directly. Consider three coordinate frames S, S' and S'', which denote the space frame with the $\hat{z}$ axis aligning with the vertical, the body frame that coincide with S at $t = 0$ and another space frame related to S by a rotation of $-\alpha$ about the $\hat{y}$ axis respectively.

From the schematic shown in Problem 1, a fixed point $\mathbf{r}'$ on the ring undergoes a rotation by $\theta = -2\pi \cos\alpha$ about the $\hat{z''}$ axis over one revolution of gyration, because the ratio of the circumference of the ring of contact to the circumference of the ring is $\cos\alpha$. We can hence relate $\mathbf{r}'$ and its representation $\mathbf{r}''$ in the S'' frame by

$$\mathbf{r}'' = U_{z''}(\theta) U_{y'}(\alpha)\, \mathbf{r}',$$

where

$$U_{y'}(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

and

$$U_{z''}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\mathbf{r}''$ and its representation $\mathbf{r}$ in the S frame is related by

$$\mathbf{r} = U_y^T(\alpha)\,\mathbf{r}'',$$

where

$$U_y(\alpha) = U_{y'}(\alpha),$$

we can represent $\mathbf{r}$ as

$$\begin{aligned} \mathbf{r} &= U_y^T(\alpha) U_{z''}(\theta) U_{y'}(\alpha)\,\mathbf{r}' \\ &= U(T)\,\mathbf{r}'. \end{aligned}$$

Therefore,

$$U(T) = \begin{bmatrix} \cos^2\alpha\cos\theta + \sin^2\alpha & -\sin\theta\cos\alpha & \sin\alpha\cos\alpha(\cos\theta - 1) \\ \sin\theta\cos\alpha & \cos\theta & \sin\theta\sin\alpha \\ \sin\alpha\cos\alpha(\cos\theta - 1) & -\sin\theta\sin\alpha & \sin^2\alpha\cos\theta + \cos^2\alpha \end{bmatrix},$$

which gives the same answer as Eq. (2) when plugging in the values of α and θ.