Extended summary of the renormalization group calculation of percolation exponents

This is to make up for an excruciatingly incoherent lecture. You have worked hard following and contributing to various parts of a major calculation, and deserve a more satisfying finale. Here it is, in the form of notes.

Parameter transformations for an infinitesimal scale change

The four parameters in the Potts Hamiltonian are the coefficient of the squared-gradient term c, the control parameter r that multiples the quadratic term, the coefficient w of the cubic term, and the parameter h that selects the Potts state. Upon changing the logarithmic scale of the system from λ to $\lambda + \delta\lambda$, these parameters are changed by a combination of three mechanisms:

1. Tracing out field modes in the highest momentum shell.
2. Simple scale factors that arise from (i) restoring the original density of modes in the momentum basis (the transformed system then being slightly larger than the original one) and (ii) the transformation of gradients.
3. Field rescaling to restore the value $c = 1$.

Here are the combined effects of these changes:

$$\begin{aligned} r(\lambda+\delta\lambda) &= \chi^2 \frac{r^<}{b^{(2-1)D}} \\ w(\lambda+\delta\lambda) &= \chi^3 \frac{w^<}{b^{(3-1)D}} \\ h(\lambda+\delta\lambda) &= \chi^1 h^<, \end{aligned}$$

where

$$\chi = \sqrt{b^{D+2}/c^<}.$$

The parameters with superscript $<$ are the coefficients just after tracing out modes in the momentum shell:

$$c^< = 1 + \left(\frac{\delta c}{\delta\lambda}\right)\delta\lambda$$

$$
\begin{aligned}
r^{<} &= r(\lambda) + \left(\frac{\delta r}{\delta\lambda}\right)\delta\lambda \\
w^{<} &= w(\lambda) + \left(\frac{\delta w}{\delta\lambda}\right)\delta\lambda \\
h^{<} &= h(\lambda).
\end{aligned}
$$

The parameter h is unchanged by the momentum-shell trace because it only couples to the zero-momentum component of the fields. Using these in the expressions for the transformed parameters and $b = 1 + \delta\lambda$, we obtain the flow equations below:

$$
\begin{aligned}
\dot{r} &= 2r - r\left(\frac{\delta c}{\delta\lambda}\right) + \left(\frac{\delta r}{\delta\lambda}\right) && (1)\\
\dot{w} &= (3 - D/2)w - \frac{3}{2}w\left(\frac{\delta c}{\delta\lambda}\right) + \left(\frac{\delta w}{\delta\lambda}\right) && (2)\\
\dot{h} &= \frac{1}{2}\left(D + 2 - \left(\frac{\delta c}{\delta\lambda}\right)\right)h. && (3)
\end{aligned}
$$

We will also need to know the net effect of rescaling on the fields. After an infinitesimal scale change the fields of the rescaled model have this relation to the original fields:

$$
\begin{aligned}
\Psi'_i(x/b) &= \frac{b^D}{\chi}\Psi_i^{<}(x) \\
&= \frac{1}{2}\left(D - 2 + \left(\frac{\delta c}{\delta\lambda}\right)\right)\Psi_i^{<}(x). && (4)
\end{aligned}
$$

Results of the cubic-term perturbation calculation

The trace over modes in the high momentum shell is performed via diagramatic perturbation theory applied to the cubic term of the Potts Hamiltonian. Below are the results of those calculations:

$$
\begin{aligned}
\frac{\delta c}{\delta\lambda} &= 18(q-2)w^2\Omega_D\left(\frac{K^D}{(K^2+r)^3} - \frac{4}{D}\frac{K^{D+2}}{(K^2+r)^4}\right) \\
\frac{\delta r}{\delta\lambda} &= -18(q-2)w^2\Omega_D\frac{K^D}{(K^2+r)^2} \\
\frac{\delta w}{\delta\lambda} &= 36(q-3)w^3\Omega_D\frac{K^D}{(K^2+r)^3}.
\end{aligned}
$$

Rationalized parameters, percolation limit, and $D = 6 - \epsilon$

In analyzing the flow equations (1,2,3) we start with the case where at the original scale, and therefore also at all larger scales, $h = 0$. This will be revisited later, when we analyze the free energy. To tidy up the equations for the flow in the (r, w) plane, we absorb some constant factors by defining "rationalized" parameters:

$$\begin{aligned} \tilde{r} &= r/K^2 \\ \widetilde{w}^2 &= \Omega_D K^{D-6} w^2. \end{aligned}$$

Along with these definitions, we make the following simplifications that apply specifically to the percolation limit of the Potts model:

1. Set $q = 1$ and $D = 6 - \epsilon$. The deviation ϵ from the upper critical dimension is treated as a small parameter and we discard higher order terms.
2. The factors $(K^2 + r)^{-n}$ are expanded for $r \ll K^2$, since we are only interested in the flow in the vicinity of the fixed point where $r = O(\epsilon)$.

Here are the resulting flow equations (accents on the rationalized parameters have been dropped):

$$\begin{aligned} \dot{r} &= 2r + 18w^2 - 30rw^2 + \cdots \\ \dot{w} &= \frac{\epsilon}{2} w - 63w^3 + \cdots \end{aligned}$$

Here $\cdots$ represents terms that are higher order in ϵ either explicitly or implicitly because they involve higher powers in r and w than the terms retained (since $r = O(\epsilon)$ and $w = O(\sqrt{\epsilon})$ near the fixed point).

Fixed point properties

The Wilson-Fisher fixed point ($\epsilon > 0$), by the flow equations above, is located at

$$\begin{aligned} r^* &= -\frac{\epsilon}{14} \\ w^* &= \sqrt{\frac{\epsilon}{126}}. \end{aligned}$$

Linearizing the flow equations about the fixed point and defining $\delta r'$ as the coordinate along the unstable axis (with origin at the fixed point), we obtain

$$\delta r'(\lambda_0 + \lambda) = \delta r'(\lambda_0) \exp{(\lambda/\nu)}, \tag{5}$$

where the fundamental renormalization group exponent ν, characterizing the approach to criticality, has the value

$$\nu = \frac{1}{2} + \frac{5}{84}\epsilon + O(\epsilon^2).$$

The second fundamental RG exponent, characterizing the critical state itself, is defined by the fixed point value of

$$\begin{aligned} \eta &= \left.\frac{\delta c}{\delta \lambda}\right|_* \\ &= -\left.6w^2\right|_* \\ &= -\frac{\epsilon}{21} + O(\epsilon^2), \end{aligned}$$

where the second line is expressed in terms of rationalized parameters as before. Substituting this definition in the flow equation (3) for h, we obtain the scaling of h when the system is tuned to the critical state ($\delta r' = 0$) and flows indefinitely to the fixed point:

$$h(\lambda_0 + \lambda) = h(\lambda_0) \exp{\left(\frac{1}{2}(D + 2 - \eta)\lambda\right)}. \tag{6}$$

Using (4), a similar rule applies to the multiplier of the fields (superscripts give the scale of the model on which the fields are defined):

$$\Psi_i^{\lambda_0+\lambda}(x/b) = \Psi_i^{\lambda_0}(x) \exp{\left(\frac{1}{2}(D - 2 + \eta)\lambda\right)}. \tag{7}$$

Exponent of the correlation function at criticality

Let

$$G^*_{ij}(x) = \langle \Psi_i^0(0)\Psi_j^0(x)\rangle$$

be the two-point correlation function of the original system ($\lambda = 0$) when r is tuned to the critical point (under RG the parameters flow indefinitely to the fixed point). Using (7), rewrite this in terms of the model and fields at scale $b = e^{\lambda}$, and define $x_0 = x/b$:

$$\begin{aligned} G^*_{ij}(x) &= b^{-D+2-\eta} \, \langle \Psi_i^{\lambda}(0)\Psi_j^{\lambda}(x/b)\rangle \\ &= \left(\frac{|x_0|}{|x|}\right)^{D-2+\eta} \langle \Psi_i^{\lambda}(0)\Psi_j^{\lambda}(x_0)\rangle. \end{aligned}$$

We fix x_0 at some small distance, say one lattice spacing, so that the expectation value will also be a fixed constant. The result gives the power law of the correlation function at criticality,

$$G^*_{ij}(x) \propto \frac{1}{|x|^{D-2+\eta}},$$

and establishes the RG exponent η as the correction relative to the pure squared-gradient model. The small epsilon calculation ($\eta < 0$) shows that the correction produces a slower decay of correlations in percolation.

Controlling criticality and divergence of lengths

At the scale of the original Potts model, the w parameter is not small and the system is far from the Wilson-Fisher fixed point where $w^* = O(\sqrt{\epsilon})$. On the other hand, a fixed (and modest) level of rescaling will flow the original parameters $(r_0 + \delta r, w_0)$ into a small enough neighborhood of the fixed point such that the linearized flow equations are a good approximation. The value $\delta r = 0$ identifies the separatrix in the flow. For $\delta r > 0$ the flow is toward large positive r, or a large scale model where Ψ has a vanishing mean value, while the large scale model favors a non-vanishing Ψ when $\delta r < 0$. Under RG flow by some fixed scale b_0 the value of δr maps to the coordinate $\delta r'$ along the unstable direction of the fixed point. Linearizing this regular map, so $\delta r' = A\delta r$, and combining with (5) we obtain the equation

$$\delta r' = b^{1/\nu} A \delta r, \tag{8}$$

after additional scaling by factor b beyond the rescaling b_0 needed to arrive at the fixed point neighborhood. The point of this equation is that it tells us, for however closely we tune the transition with δr, how large a rescaling b is required to flow $\delta r'$ some fixed distance $\delta r'_0 > 0$ from the fixed point. Keeping in mind that $\delta r'$ can have either sign, we obtain

$$b = B|\delta r|^{-\nu}, \tag{9}$$

where B combines the constants A and δr_0. Models that flow to $\pm\delta r'_0$ are (up to sign) essentially unique because the w parameter is attracted to the stable axis of the fixed point. These models therefore have identical characteristics (again up to sign). If the feature is a length, such as the decay length of the (non-critical) correlation function, we must remember that this length should be multiplied by the rescaling factor b if it is to describe the same feature in the original model. Equation (9) therefore gives the power law for the divergence of lengths in the model.

Free energy and percolation exponents

The free energy of the Potts model has a singular contribution that depends on the variables δr and h because the signs of these control the transition in the mean value of the field Ψ. We therefore write

$$F = F_0 + F_1(\delta r, h),$$

where F_0 may be taken as a constant in the domain of interest where both δr and h are small. Using (8) and (6) we can rewrite the singular contribution in terms of the model at scale b:

$$F_1(\delta r, h) = \frac{1}{b^D} F_1(b^{1/\nu}\ \delta r, b^{(D+2-\eta)/2} h)$$

The factor b^D corrects for the volume increase of the model at scale b. To obtain the magnetization and susceptibility we take respectively one and two derivatives with respect to h, set $h = 0$, and as in the discussion of diverging lengths, select b such that

$$b^{1/\nu}\ \delta r = \pm \delta r_0$$

for some fixed $\delta r_0 > 0$. The result of those calculations is

$$\begin{aligned} m &= \left(\frac{|\delta r|}{\delta r_0}\right)^{\nu(D-2+\eta)/2} \partial_h F_1(\pm\delta r_0, 0) \\ \chi &= \left(\frac{\delta r_0}{|\delta r|}\right)^{\nu(2-\eta)} \partial_h^2 F_1(\pm\delta r_0, 0), \end{aligned}$$

and identifies the following combination of RG exponents as the critical exponents of percolation:

$$\begin{aligned} \beta &= \nu(D-2+\eta)/2, \\ \gamma &= \nu(2-\eta). \end{aligned}$$

Finally, substituting $D = 6 - \epsilon$ and the result of the small epsilon calculation of ν and η,

$$\begin{aligned} \beta &= 1 - \frac{\epsilon}{7} + O(\epsilon^2), \\ \gamma &= 1 + \frac{\epsilon}{7} + O(\epsilon^2). \end{aligned}$$