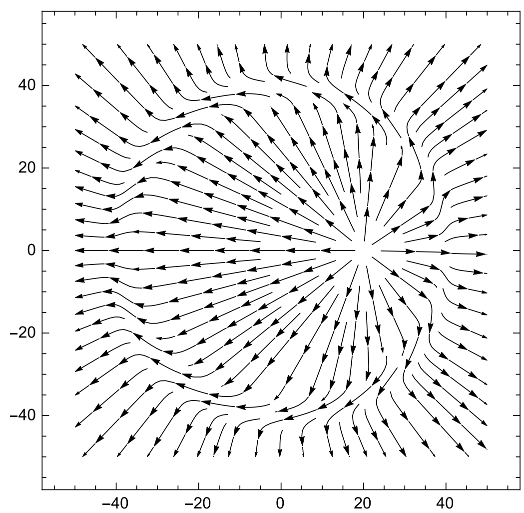
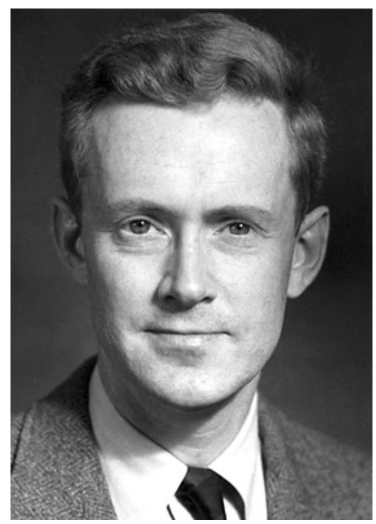
Assignment 9

Due date: Tuesday, November 7[1]

Purcell's iconic radiation sketch

In this problem you will recreate a numerically accurate rendition of the sketch in Edward Purcell's famous textbook showing how the continuity of electric field lines, produced when a charge transitions between two states of uniform motion, explains the transverse nature and magnitude enhancement — over static fields — of radiation fields.





As in Purcell's model, the charge moves along a straight line, which for us defines the x-axis. For the position we take the convenient function

$$x(t) = v \log\left(1 + e^{t}\right),$$

where t is the time in our frame of reference and v is the asymptotic velocity at large positive t. Since the asymptotic velocity at large negative t is zero, this describes a transition between two states of uniform motion. The approach to these asymptotic states is exponentially fast. When properly executed our electric field plot should therefore display, as above, the simple electric fields of the uniform states of motion.

[1] You may continue to set $c = 1$ in this assignment.

Make a plot of the electric field in the x-y plane at time $t = 40$ and in the range $-50 < x < 50$, $-50 < y < 50$. Set $v = 1/2$, so that at time $t = 40$ the particle will be at $x = 20$. All your numerical computations will be based on the formula for $\mathbf{E}$ you derived in the last assignment. First establish that the direction of $\mathbf{E}$ also lies in the x-y plane. For the most part you only need to carefully transcribe the symbols in the formula into statements of computer code (I used *Mathematica*). However, one part of the computation — determining the retarded time of the source — will probably be a new challenge. I suggest you define a function $\mathsf{tret}[t, x, y]$ that returns the retarded time at any position (x, y) and time t (your plot will be for $t = 40$). If you do not have access to a package that provides for simple root-finding you can code this by hand using bisection. In any case, be careful that your function returns the retarded-time solution rather than the advanced-time solution! To speed up your code make only one call to $\mathsf{tret}[t, x, y]$ at each (x, y) and use it in all the quantities $\boldsymbol{\beta}$, $\hat{\mathbf{n}}$, etc., that require it.

A mysterious vector density

The midterm exam featured the following scalar (rotational invariant) constructed from the 3-vector potential:

$$V^0 = \mathbf{A} \cdot \nabla \times \mathbf{A}.$$

1. As the notation suggests, V^0 is the time-component of a 4-vector V^α. Write a 4-vector definition of V^α in terms of some combination of (4-vectors/tensors) A, F and $\tilde{F}$ that reduces to V^0 for $\alpha = 0$.
2. Show that V^α has vanishing 4-divergence for the most general electromagnetic field produced by a single point charge. *Hint*: Consult the previous assignment on the value of $\mathbf{E} \cdot \mathbf{B}$.

   From this property we know that

   $$M = \int d^3x \, V^0$$

   is constant in time (subject to the usual boundary assumptions) — a conserved quantity! In the midterm you showed that M is gauge invariant (again, up to boundary behavior assumptions).
3. Take up the following plan of action to determine what M might be, and what scale factor should be used in its definition.
   (a) Construct the 3-vector potentials $\mathbf{A}_\pm$ for pure modes of right and left circularly polarized light. You will need to multiply your modes by a very slowly varying scalar "envelope-function" to normalize them (this function is arbitrary in all other respects).

(b) Calculate the total energies $\mathcal{E}$ of your modes (by integrating θ^{00}) and fix the normalization by the property that $\mathcal{E} = \hbar\omega$, where ω is the mode frequency.

(c) Keeping the same normalization, calculate the M-values of your modes. Use these results to motivate a choice of scale for the mystery property M.