Equilibrium properties of the point-particle gas in three dimensions

1. The bulk modulus is related to the energy by $B = V\frac{d^2E}{dV^2}$. Substituting in $E(V) = E(V_0)\left(\frac{V_0}{V}\right)^{\frac{2}{3}}$, we have $\frac{d^2E}{dV^2} = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)E(V_0)\frac{V_0^{\frac{2}{3}}}{V^{\frac{2}{3}+2}} = \frac{10}{9}\frac{E(V_0)\left(\frac{V_0}{V}\right)^{\frac{2}{3}}}{V^2} = \frac{10}{9}\frac{E(V)}{V^2}$, so $B(V) = \frac{10}{9}\frac{E(V)}{V}$, or
$$B(V)V = \frac{10}{9}E(V)$$
2. The speed is given by
$$v_s = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{B}{(Nm/V)}} = \sqrt{\frac{B(V)V}{Nm}}$$
3. Using the result from part 1, we have $v_s = \sqrt{\frac{10}{9}\frac{E(V)}{Nm}}$. Using $E = \frac{1}{2}Nm\langle v_x^2 + v_y^2 + v_z^2\rangle$, we get
$$v_s = \sqrt{\frac{5}{9}\langle v_x^2 + v_y^2 + v_z^2\rangle} = \frac{\sqrt{5}}{3}v_{rms},$$
Which means $\beta = \frac{\sqrt{5}}{3}$.

Displacement, density, and pressure in sound

We begin by noting that $\nabla \cdot \boldsymbol{s}$ being positive means there is a net displacement outwards from the volume, meaning particles are moving out of the volume, and therefore the density in the volume and the pressure is decreasing. This means the signs must be negative.

Furthermore, $\nabla \cdot \boldsymbol{s}$ is unitless, because $\boldsymbol{s}$ is a displacement, so it has units of distance, and ∇ is a spatial derivative and therefore has units of inverse distance. The quantity $\frac{\delta n}{n}$ is also trivially unitless. For the quantity $\frac{\delta p}{B}$, we recall $B = V\frac{\partial p}{\partial V}$, which has units of p, so $\frac{\delta p}{B}$ is also unitless.

Consider a set of particles in a small volume V whose lower left corner is initially at (x, y, z), and initially has dimensions $\Delta x, \Delta y, \Delta z$. The leftmost face undergoes a displacement $s_x(x, y, z)$, and the rightmost face undergoes a displacement $s_x(x + \Delta x, y, z)$. The new x coordinates of the left and right faces are then $x + s_x(x, y, z)$ and $x + \Delta x + s_x(x + \Delta x,\ y,\ x)$, so the new dimension along x is $\Delta x + s_x(x + \Delta x,\ y,\ x) - s_x(x, y, z)$, and the change in volume due to this displacement is $\Delta y\Delta z\big(s_x(x + \Delta x,\ y,\ x) - s_x(x, y, z)\big) = \Delta y\Delta z\Delta x\frac{(s_x(x+\Delta x, y, x) - s_x(x, y, z))}{\Delta x} \approx V\frac{\partial s_x}{\partial x}$. We can repeat this argument for the other dimensions, and find that the total change in volume is $\delta v = V\left(\frac{\partial s_x}{\partial x} + \frac{\partial s_y}{\partial y} + \frac{\partial s_z}{\partial z}\right) = V\ \nabla \cdot \boldsymbol{s}$, or $\nabla \cdot \boldsymbol{s} = \frac{\delta V}{V}$. Multiplying and dividing by δp, we find $\nabla \cdot \boldsymbol{s} = \frac{1}{V}\frac{\delta V}{\delta p}\ \delta p = \frac{\delta p}{V\frac{\delta p}{\delta V}}$, and using $B = -V\frac{\partial p}{\partial V}$ we finally obtain
$$\nabla \cdot \boldsymbol{s} = -\frac{\delta p}{B}.$$

Again consider a set of particles in a small volume V whose lower left corner is at (x, y, z), and has dimensions $\Delta x, \Delta y, \Delta z$. As the particles undergo a displacement, the number of particles that exit the volume crossing the left face is $-n\,\Delta y\Delta z S_x(x, y, z)$ and the number of particles that exit crossing the right face is $n\,\Delta y \Delta z S_x(x+\Delta x, y, z)$. The number of particles that exit the volume by crossing the right and left faces is then $n\Delta y\Delta z\big(s_x(x+\Delta x,\ y,\ x) - s_x(x,y,z)\big) =$ $n\Delta y\Delta z\Delta x\,\frac{(s_x(x+\Delta x,y,x)-s_x(x,y,z))}{\Delta x} \approx nV\,\frac{\partial s_x}{\partial x}$. We can repeat this argument for the other dimensions and find that the total number of particles that exit the volume is $nV\left(\frac{\partial s_x}{\partial x}+\frac{\partial s_y}{\partial y}+\frac{\partial s_z}{\partial z}\right) = nV\ \nabla\cdot\boldsymbol{s}$. The change in number of particles in the box is then $\Delta N = \ -nV\ \nabla\cdot\boldsymbol{s}$, and negative sign is because the righthand expression is the number of particles that have exited the volume. This can be rewritten as $-\frac{\left(\frac{\Delta N}{V}\right)}{n} = \nabla\cdot\boldsymbol{s}$, but $\frac{\Delta N}{V}$ is the change in density, giving finally

$$-\frac{\delta n}{n} = \nabla\cdot\boldsymbol{s}.$$

Note that we could have obtained this from $\nabla\cdot\boldsymbol{s} = \frac{\delta V}{V}$ as follows: $\delta n = \frac{\partial n}{\partial V}\delta V = \left(\frac{\partial}{\partial V}\frac{N}{V}\right)\delta V =$ $-\frac{N}{V^2}\delta V = -n\frac{\delta V}{V} = \ -n\ \nabla\cdot\boldsymbol{S}$, and dividing by $-n$ we obtain the desired expression. In this derivation, however, the physics represented by the equation is not as clear.

The wave equation in three dimensions

1. $\nabla^2 p = \frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}$. If p varies spatially only in x, $\frac{\partial p}{\partial z} = \frac{\partial p}{\partial y} = 0$, so $\frac{\partial^2 p}{\partial z^2} = \frac{\partial^2 p}{\partial y^2} = 0$. With this, we have $\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2} = \frac{\partial^2 p}{\partial x^2}$, and $\frac{\partial^2 p}{\partial t^2} = v^2\nabla^2 p$ reduces to $\frac{\partial^2 p}{\partial t^2} = v^2\frac{\partial^2 p}{\partial x^2}$.
2. Plugging $p_0\cos(kx-\omega t)$ into $\frac{\partial^2 p}{\partial t^2} = v^2\frac{\partial^2 p}{\partial x^2}$, the left-hand side becomes $-\omega^2 p_0\cos(kx-\omega t)$ and the right-hand side becomes $-v^2k^2p_0\cos(kx-\omega t)$, which means $\omega^2 = v^2k^2$, or
$$\omega = vk.$$
Next, we note that $p_0\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) = p_0\cos\big(k_x x + k_y y + k_z z - \omega t\big)$. Recalling $\nabla^2 p =$ $\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}$, we see
$$\begin{aligned}\nabla^2\cos&\big(k_x x + k_y y + k_z z - \omega t\big)\\ &= -k_x^2 p_0\cos\big(k_x x + k_y y + k_z z - \omega t\big) - k_y^2 p_0\cos\big(k_x x + k_y y + k_z z - \omega t\big)\\ &\quad - k_z^2 p_0\cos\big(k_x x + k_y y + k_z z - \omega t\big)\\ &= -\big(k_x^2+k_y^2+k_z^2\big)p_0\cos\big(k_x x + k_y y + k_z z - \omega t\big)\\ &= -\boldsymbol{k}\cdot\boldsymbol{k}\ p_0\cos\big(k_x x + k_y y + k_z z - \omega t\big)\end{aligned}$$
The second derivative with respect to time is $-\omega^2 p_0\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)$, which means
$$v^2\boldsymbol{k}\cdot\boldsymbol{k} = v^2|\boldsymbol{k}|^2 = \omega^2.$$
3. The fact that the medium is isotropic implies rotational symmetry, meaning that the equation should remain unchanged by rewriting it in terms of coordinates rotated relative to the initial ones. In particular, changing $x\to y, y\to z, z\to x$ should leave the equation unchanged, which is true for $\frac{\partial^2 p}{\partial t^2} = v^2\nabla^2 p$.
4. In this case $\nabla^2 p = \frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2} = \ -k_x^2 p_0 e^{-k_y y}\cos(k_x x-\omega t) + k_y^2 p_0 e^{-k_y y}\cos(k_x x-\omega t) =$ $-\big(k_x^2-k_y^2\big)p_0 e^{-k_y y}\cos(k_x x-\omega t)$, and $\frac{\partial^2 p}{\partial t^2} = -\omega^2 p_0 e^{-k_y y}\cos(k_x x-\omega t)$. This means
$$v^2\big(k_x^2-k_y^2\big) = \omega^2$$

Sound modes in an organ pipe

1. At the $x = 0$ the pipe is closed by a solid barrier, so there is no displacement there. The boundary condition is therefore

$$s(0, t) = 0.$$

2. Recall that $\Delta p \propto \frac{\partial s}{\partial x}$, so at $x = L$, we have

$$\frac{\partial s}{\partial x}(L, t) = 0.$$

3. The normal modes will be of the form $A\cos(\omega t)\sin(kx)$, because $\sin(k0) = 0$, as required by the boundary conditions. The second boundary condition p$\frac{\partial s}{\partial x}(L, t) \propto \cos(kL) = 0$ implies $kL = \frac{2\pi L}{\lambda} = \left(n + \frac{1}{2}\right)\pi$, or $\lambda = \frac{2L}{n+\frac{1}{2}}$. In air, sound has a linear dispersion relation, so $\omega = v_s k$ which gives $f = \frac{v_s}{\lambda} = v_s \frac{\left(n+\frac{1}{2}\right)}{2L}$. Solving for L we obtain $L = \frac{v_s}{f}\frac{n+\frac{1}{2}}{2}$. For the lowest mode, $n = 0$, so

$$L = \frac{v_s}{4f} = \frac{340}{4 \cdot 440}\,\mathrm{m} \approx 0.19\ \mathrm{m}.$$

4. Substituting $L = \frac{v_s}{4f_0}$ into $f = v_s \frac{\left(n+\frac{1}{2}\right)}{2L}$, where we've used f_0 to refer to the lowest mode, we get $f = 2f_0\left(n + \frac{1}{2}\right)$. For the next lowest, $n = 1$, we have

$$f_1 = 2f_0\left(1 + \frac{1}{2}\right) = 3f_0 = 1320\ \mathrm{Hz}.$$