Homework 8

Due date: Wednesday, March 27

Equilibrium properties of the point-particle gas in three dimensions

1. Starting with the result you obtained on the previous homework (total energy in a 3D gas of point particles)

$$E(V) = E(V_0)\left(\frac{V_0}{V}\right)^{2/3},$$

show that

$$B(V)V = \alpha\, E(V),$$

where $B(V)$ is the bulk modulus of the gas and α is a numerical constant whose value you need to determine.

2. Express the sound speed in the gas, v_s, in terms of V, $B(V)$, m, and N, the number of particles in the gas. You do not need the result of part (1) for this.

3. Using the expression

$$E(V) = N\,\frac{1}{2}m\langle v_x^2 + v_y^2 + v_z^2\rangle,$$

where the angle brackets denote "average over all particles", show that the sound speed v_s is proportional to the root-mean-square particle speed defined as

$$v_{\mathrm{rms}} = \sqrt{\langle v_x^2 + v_y^2 + v_z^2\rangle}.$$

In other words, show that

$$v_s = \beta\, v_{\mathrm{rms}}.$$

and determine the numerical constant β. For this part you need to use the results of parts (1) and (2).

Displacement, density, and pressure in sound

A medium is comprised of particles with number density (number per unit volume) n, and has bulk modulus B. Sound in the medium is described by a time-dependent vector field $\mathbf{s}(\mathbf{r}, t)$ that specifies the displacement of large groups of particles (large compared to the mean-free-path, small compared to the sound wavelength). In the presence of sound, there are small perturbations of the density, $\delta n(\mathbf{r}, t)$, and the pressure, $\delta p(\mathbf{r}, t)$. By using the hints provided below, show that these are related to the displacement as follows[1]:

$$\pm \frac{\delta n}{n} = \nabla \cdot \mathbf{s} = \pm \frac{\delta p}{B}$$

- Using intuition and basic calculus, first determine the correct signs!
- Confirm that the units are correct!
- For one relation you will want to consider a **fixed set of particles** moving in such a way that the volume they occupy changes by δv.
- For another relation you will want to consider a **fixed volume** in space and the change in the number of enclosed particles due to their flow through the surface of the volume.
- As in the derivation of the local relationship for charge conservation, you will need the divergence theorem of vector calculus:

$$\int_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = \int_{\mathcal{V}} \nabla \cdot \mathbf{v} \, d^3\mathbf{r} \, .$$

The wave equation in three dimensions

To a very good approximation, the medium of sound (air) is 3D and isotropic. In lecture our particle displacements $s(x, t)$ only varied with x and were always along x as well. The "wave equation" we derived had the same preference for x :

$$\frac{\partial^2 s}{\partial t^2} = v^2 \frac{\partial^2 s}{\partial x^2} \, .$$

Applying $\partial/\partial x$ to this equation, recognizing $\partial s/\partial x$ as $\nabla \cdot \mathbf{s}$ when $\mathbf{s}$ only has an x-component, and using the previous problem, the wave equation can be expressed as an equation for the pressure, a scalar:

$$\frac{\partial^2 p}{\partial t^2} = v^2 \frac{\partial^2 p}{\partial x^2} \, . \tag{1}$$

[1] In this problem and the next we are using the convention where vectors are rendered in bold-face font. **Please keep vectors and scalars distinct in your solutions as well**.

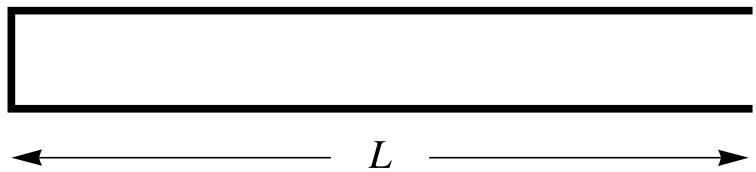
In this problem you will gain some familiarity with the generalization of this equation that describes sound in a 3D isotropic medium:

$$\frac{\partial^2 p}{\partial t^2} = v^2 \nabla^2 p \tag{2}$$

1. Show that (2) reduces to (1) when p varies (in space) only along x.
2. Suppose
$$p(x,t) = p_0 \cos(kx - \omega t)$$
is a solution of (1). What is the relationship between k and ω? Next consider the generalization
$$p(\mathbf{r},t) = p_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \ ,$$
a wave running in the direction of the vector $\mathbf{k}$. Show that this pressure satisfies (2). What is the relationship between $\mathbf{k}$ and ω? This form of the pressure may help you understand the operator relationship $\nabla \cdot \nabla = \nabla^2$.
3. What symmetry property supports (2) as the correct generalization? A simple statement (no math) suffices here!
4. Suppose the pressure has the partially evanescent form
$$p(\mathbf{r},t) = p_0 e^{-k_y y} \cos(k_x x - \omega t) \ ,$$
where k_x and k_y are real numbers. What is the relationship between k_x, k_y and ω ?

Sound modes in an organ pipe

Take as a simple model of an organ pipe a cylinder with one end closed and the other end open:



1. When describing the sound in the cylinder in terms of the displacement $s(x,t)$ of cylindrical volumes of gas relative to equilibrium, what boundary condition should be imposed on $s(x,t)$ at the closed end, $x = 0$?
2. The appropriate boundary condition at the open end, $x = L$, is that the pressure amplitude Δp vanishes[2]. What boundary condition does that impose on $s(x,t)$?
3. Suppose you are building an organ pipe whose lowest tone (normal mode) has frequency f (cycles per second). What should be the length L for $f = 440$ Hz, assuming the standard sound speed 340 m/s?
4. A pipe tuned to 440 Hz as its lowest tone will also resonate at higher frequencies. What is the next lowest?

[2]This is not exactly true but a very good approximation. There is in fact a sound wave that propagates out of the open end — the very same sound we hear — but whose pressure amplitude even a short distance from the open end is very much smaller than the pressure inside the pipe.