Assignment 8

Due date: Halloween[1]

Charged particle motion from the action principle

In this problem we treat the electromagnetic field as given (produced by an external agent) and study its effect on the motion of a relativistic particle of mass m and charge q. The converse, or electromagnetic field produced by a given charged-particle world line, gets equal treatment in the second problem.

So as to not confuse a general event x in space-time with the particle world-line, we use the notation $\xi(t)$ for the latter, where t is an arbitrary parameter (not necessarily time). Consider the following definition of the 4-current density in space-time associated with the particle:

$$J^\alpha(x) = q \int (\mathcal{L}dt) u^\alpha(t)\, \delta^4(x - \xi(t)),$$

where

$$d\tau = \mathcal{L}dt = \sqrt{\dot{\xi}^\alpha \dot{\xi}_\alpha}\, dt$$

is the proper-time element and $u^\alpha = \dot{\xi}^\alpha / \mathcal{L}$ is the 4-velocity.

1. Confirm that this geometrical definition matches the usual definition of the 4-current density by choosing time $t = x^0$ as the arbitrary parameter and explicitly evaluating the integral. Specifically, show that

$$\begin{aligned} J^0(t, \mathbf{x}) &= \rho(t, \mathbf{x}) = q\, \delta^3(\mathbf{x} - \boldsymbol{\xi}(t)) \\ \mathbf{J}(t, \mathbf{x}) &= q\mathbf{v}(t)\, \delta^3(\mathbf{x} - \boldsymbol{\xi}(t)). \end{aligned}$$

2. Return to the integral expression for $J^\alpha(x)$ without the specialization $t = \text{time}$. Show that

$$S_{\text{int}}[\xi] = \int d^4x\, J^\alpha(x) A_\alpha(x) = q \int dt\, \dot{\xi}^\alpha(t) A_\alpha(\xi(t)).$$

Here S_{int} is the term in the action that couples the particle to the electromagnetic field.

[1] As a special "treat", we set $c = 1$ in this assignment.

3. Obtain the equations of motion for the particle from the combined action,

$$S[\xi] = S_{\text{free}}[\xi] + S_{\text{int}}[\xi],$$

where

$$S_{\text{free}}[\xi] = m \int \mathcal{L} dt$$

is the action of the free particle (derived in lecture) with a particular choice of scale factor. Specifically, show that the Euler-Lagrange equations imply

$$m\dot{u}^\alpha(t) = q\, F^{\alpha\beta}(\xi(t))\, \dot{\xi}_\beta(t). \tag{1}$$

4. Again using time as the parameter, express the four components of the equations of motion (1) in terms of $\mathbf{E}$ and $\mathbf{B}$, and the components

$$\begin{aligned} mu^\alpha &= (\mathcal{E}, \mathbf{p}) \\ \dot{\xi}^\alpha &= (1, \mathbf{v}). \end{aligned}$$

Field produced by a relativistic charged particle

Consider a particle of charge q described by a general world-line $\xi(\tau)$, parameterized by the proper time τ elapsed along the world-line. In lecture we derive the formula[2]

$$\partial^\alpha A^\beta(x) = \left(\frac{q}{y \cdot u}\right) \frac{d}{d\tau} \left(\frac{y^\alpha u^\beta}{y \cdot u}\right)\Bigg|_{\tau=\tau_0}$$

for the gradient of the 4-vector potential produced by the particle. As in lecture, $u(\tau)$ is the particle 4-velocity and $y(\tau) = x - \xi(\tau)$ is the 4-vector separation between x and events along the world line. At the parameter value $\tau = \tau_0$, the separation y is null and the source $\xi(\tau_0)$ lies on the *past* light-cone of x.

1. From the above formula for $\partial^\alpha A^\beta$ obtain $F^{\alpha\beta}$ and show that

$$\begin{aligned} \mathbf{E} &= \frac{q}{y \cdot u}\left(\frac{\mathbf{R}a^0 - R\mathbf{a}}{y \cdot u} + \frac{\mathbf{R}u^0 - R\mathbf{u}}{(y \cdot u)^2}(1 - y \cdot a)\right) \\ \mathbf{B} &= \hat{\mathbf{n}} \times \mathbf{E}. \end{aligned}$$

Here $(a^0, \mathbf{a}) = du^\alpha/d\tau$ are the components of the 4-acceleration and the null separation 4-vector is expressed in terms of a distance R and unit 3-vector as $y^\alpha = (R, \mathbf{R}) = (R, R\,\hat{\mathbf{n}})$.

[2] 4-vector dot products, such as $y \cdot u$, are shorthand for $y^\gamma u_\gamma$.

2. Rewrite the expression for $\mathbf{E}$ in terms of the ordinary 3-velocity $\boldsymbol{\beta}$, the 3-acceleration $\dot{\boldsymbol{\beta}} = d\boldsymbol{\beta}/dt$, $\gamma = 1/\sqrt{1-\beta^2}$ and arrive at:

$$\mathbf{E} = \frac{q}{(\gamma R)^2}\,\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{(1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3} + \frac{q}{R}\,\frac{\hat{\mathbf{n}} \times (\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}}{(1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3}.$$

Exercise from The Lost Jackson Codex, Vol. XIV

For any field point x not on the world-line $\xi(t)$ of a (non-tachyonic) particle, show that there is a unique t_0 and $y(t_0) = x - \xi(t_0)$ such that $y(t_0)\cdot y(t_0) = 0$ and $y^0(t_0) > 0$.