Assignment 7

Due date: Tuesday, October 24

Levi-Civita tensor

The standard Lorentz transformation rule — one Λ per index — also applies to the Levi-Civita tensor:

$$(\epsilon')^{\alpha'\beta'\gamma'\delta'} = \Lambda^{\alpha'}{}_{\alpha}\,\Lambda^{\beta'}{}_{\beta}\,\Lambda^{\gamma'}{}_{\gamma}\,\Lambda^{\delta'}{}_{\delta}\,\epsilon^{\alpha\beta\gamma\delta}.$$

Show that up to overall sign, the Levi-Civita tensor is Lorentz invariant, i.e. $\epsilon' = \pm\epsilon$.

Hints:

- Express the determinant of a general 4×4 matrix $A^{\alpha}{}_{\beta}$ in terms of Greek indices and ϵ.
- $\det AB = \det A \det B$
- $\Lambda^{\alpha}{}_{\beta}\,\Lambda_{\alpha}{}^{\gamma} = \delta_{\beta}{}^{\gamma}$.

The field tensor and its dual

1. Express the Lorentz scalars
$$F^{\alpha\beta}F_{\alpha\beta}, \quad F^{\alpha\beta}\tilde{F}_{\alpha\beta}, \quad \tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta},$$
in terms of $\mathbf{E}$ and $\mathbf{B}$.
2. Curious fact:
$$F^{\alpha\beta}\tilde{F}_{\alpha\beta} = \partial_{\gamma}V^{\gamma},$$
for some 4-vector quantity V^{γ}. Find V^{γ}.
3. Is V^{γ} gauge invariant?
4. Suppose we modified the action of the electromagnetic field as follows:
$$S[A] = \int d^4x \left(\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + \lambda F^{\alpha\beta}\tilde{F}_{\alpha\beta}\right),$$
with some non-zero parameter λ. How would Maxwell's equations be changed? You can answer this question without much work if you take advantage of item 2 above.
5. Construct all possible cubic Lorentz invariants from F and $\tilde{F}$.