Assignment 7

Due date: Tuesday, October 24

Levi-Civita tensor

The standard Lorentz transformation rule — one \( \Lambda \) per index — also applies to the Levi-Civita tensor:

\[
(\epsilon')^{\alpha'\beta'\gamma'\delta'} = \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta} \Lambda^{\gamma'}_{\gamma} \Lambda^{\delta'}_{\delta} \epsilon^{\alpha\beta\gamma\delta}.
\]

Show that up to overall sign, the Levi-Civita tensor is Lorentz invariant, i.e. \( \epsilon' = \pm \epsilon \).

Hints:

- Express the determinant of a general \( 4 \times 4 \) matrix \( A_{\alpha\beta} \) in terms of Greek indices and \( \epsilon \).
- \( \det AB = \det A \det B \)
- \( \Lambda^{\alpha}_{\alpha'} \Lambda^{\gamma}_{\gamma'} = \delta^\gamma_{\gamma'} \).

The field tensor and its dual

1. Express the Lorentz scalars

\[
F^{\alpha\beta} F_{\alpha\beta}, \quad F^{\alpha\beta} \tilde{F}_{\alpha\beta}, \quad \tilde{F}^{\alpha\beta} \tilde{F}_{\alpha\beta},
\]

in terms of \( E \) and \( B \).

2. Curious fact:

\[
F^{\alpha\beta} \tilde{F}_{\alpha\beta} = \partial_\gamma V^\gamma,
\]

for some 4-vector quantity \( V^\gamma \). Find \( V^\gamma \).

3. Is \( V^\gamma \) gauge invariant?

4. Suppose we modified the action of the electromagnetic field as follows:

\[
S[A] = \int d^4x \left( \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + \lambda \tilde{F}^{\alpha\beta} \tilde{F}_{\alpha\beta} \right),
\]

with some non-zero parameter \( \lambda \). How would Maxwell’s equations be changed? You can answer this question without much work if you take advantage of item 2 above.

5. Construct all possible cubic Lorentz invariants from \( F \) and \( \tilde{F} \).