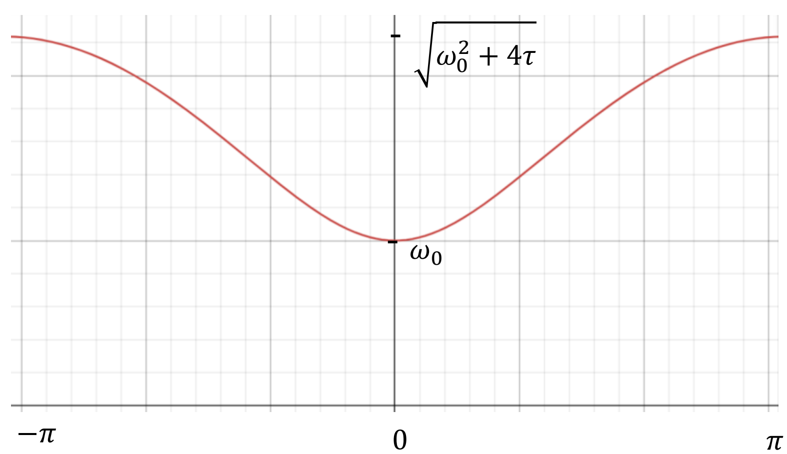
Velocity of a relativistic particle

The apparent paradox comes from treating a particle as a single wave mode, which would imply no uncertainty in the momentum of the particle, and therefore infinite uncertainty in the position of the particle, but localization is a defining characteristic of a particle. A particle is therefore a wave packet composed of many different modes, so its speed is better thought of as the group velocity of the wave packet. In fact, we can compute what this is as $v = \frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k}\sqrt{m^2 + k^2} = \frac{k}{\sqrt{m^2+k^2}} = \frac{k}{\omega} = \frac{P}{E}$, exactly the same as obtained from relativity.

Coupled pendulum dispersion relation

1. Ignoring the second term on the right, we have a differential equation describing the motion of a single pendulum (in the small amplitude limit), whose solutions are given by sines and cosines with angular frequency ω_0, so ω_0 is the natural frequency of oscillation of a single pendulum. Ignoring the first term on the right, we have a discrete version of the wave equation with waves propagating at "speed" $\sqrt{\tau}$ (in units of number of oscillators per time). τ is therefore the "speed" of propagation of a pulse through the system of coupled oscillators.
2. Substituting in equation 2 into equation 1 on the homework sheet we find $-A\omega^2 \cos(\omega t)\cos(kn) = -\omega_0^2 A\cos(\omega t)\cos(kn) + \tau A\cos(\omega t)\left(\cos(kn-k) - 2\cos(kn) + \cos(kn+k)\right)$. Dividing by $A\cos(\omega t)$ and using $\cos(kn-k) + \cos(kn+k) = 2\cos(kn)\cos(k)$ we obtain $-\omega^2\cos(kn) = \omega_0^2\cos(kn) + 2\tau\cos(kn)\left(\cos(k) - 1\right)$. Dividing by $-\cos(kn)$ we have that $\omega_0^2 A\cos(\omega t)\cos(kn)$ is a solution to equation 1 is $\omega^2 = \omega_0^2 - 2\tau(\cos(k) - 1)$, or
$$\omega = \sqrt{\omega_0^2 + 2\tau(1 - \cos(k))}$$
3. Using the trigonometric identities for sums of angles we have that $\cos(\alpha + m2\pi) = \cos(\alpha)\cos(m2\pi) - \sin(\alpha)\sin(m2\pi)$. But $\cos(m2\pi) = 1, \sin(m2\pi) = 0$ for integer m. With this, we have that $\cos(k_2 n) = \cos(k_1 n + 2\pi nK) = \cos(k_1 n)$. Therefore,
$$A\cos(\omega t)\cos(k_2 n) = A\cos(\omega t)\cos(k_1 n),$$
So $k_2 = k_1 + 2\pi K$ is the same mode as k_1.
4. We know $0 \le (1 - \cos(k)) \le 2$, and $1 - \cos(k) = 0$ for $k = 0$, and $1 - \cos(k) = 2$ for $k = \pm\pi$. From this, we have $\omega_0^2 \le \omega_0^2 + 2\tau(1 - \cos(k)) \le \omega_0^2 + 4\tau$, and since $\sqrt{x}$ is strictly increasing (assuming x is real), we have
$$\omega_0 \le \omega(k) \le \sqrt{\omega_0^2 + 4\tau},$$
and
$$\omega(0) = \omega_0, \omega(\pm\pi) = \sqrt{\omega_0^2 + 4\tau}.$$



Organ pipe, both ends open

1. For the normal mode solutions, we assume $p(x,t) = \cos(\omega t)\, f_\omega(x)$. Substituting this into the wave equation, we find $-\omega^2 \cos(\omega t)\, f_\omega(x) = v^2 \cos(\omega t)\, f''_\omega(x)$, or, simplified, $f''_\omega(x) = -\frac{\omega^2}{v^2} f_\omega(x)$, which implies $f_\omega(x) = A\cos(kx) + B\sin(kx)$, with $k = \frac{\omega}{v}$. The general solution is therefore
$$p(x,t) = \cos(\omega t)\left(A\cos\left(\frac{\omega}{v}x\right) + B\sin\left(\frac{\omega}{v}x\right)\right)$$
2. The condition $\frac{\partial p}{\partial x}(0,\ t) = \frac{\omega}{v} B\cos(\omega t) = 0$ implies $B = 0$, and therefore $p(x,t) = A\cos(\omega t)\cos\left(\frac{\omega}{v}x\right)$. The condition $\frac{\partial p}{\partial x}(L,\ t) = -\frac{\omega}{v} A\cos(\omega t)\sin\left(\frac{\omega}{v}L\right) = 0$ implies $\sin\left(\frac{\omega}{v}L\right) = 0$, and therefore $\frac{\omega}{v}L = n\pi$. This gives
$$p(x,t) = A\cos\left(\frac{n\pi v}{L}t\right)\cos\left(\frac{n\pi}{L}x\right)$$
3. From the condition $\frac{\omega}{v}L = n\pi$ we have $\omega = \frac{n\pi v}{L}$, which means
$$f = \frac{\omega}{2\pi} = \frac{nv}{2L}, n \in \mathbb{Z}$$

Point particle colliding elastically with a moving wall

1. From the moving frame's view, the lab is moving with velocity $-V_x$, and since velocities add (as vectors), the velocity of the particle in the moving frame is $\boldsymbol{v}' = \boldsymbol{v} - V_x$, which means
$$v'_x = v_x - V_x.$$
2. In the frame of the wall, the particle will recoil with the same speed perpendicular to the wall as the particle's speed before the collision, but in the opposite direction. That is,
$$v''_x = -v'_x = V_x - v_x.$$
3. Once again, using velocity addition, we have $v'''_x = v''_x + V_x$. Substituting our expression from part 2, we have
$$v''_x = 2V_x - v_x.$$