Homework 6

Due date: Wednesday, March 13

Velocity of a relativistic particle

In Physics 1116 (or 2216) you learned that the energy, rest mass and momentum of a particle are related as (units where $c = 1$)

$$E = \sqrt{M^2 + P^2} \ .$$

Then, in this course you were told that E and P have wave counterparts $\omega = E/\hbar$, $k = P/\hbar$, where $\hbar$ is Planck's constant divided by 2π. Defining the constant $m = M/\hbar$, the kinematic relationship above looks like a dispersion relation:

$$\omega(k) = \sqrt{m^2 + k^2} \ .$$

Another thing you learned in 2218 is that the speed of a wave is

$$v = \frac{\omega}{k} = \frac{E}{P} \ .$$

But in 1116 / 2216 you also learned that

$$E = m\gamma \qquad\qquad P = m\gamma v \ ,$$

where $\gamma = 1/\sqrt{1 - v^2}$, and therefore

$$v = \frac{P}{E} \ .$$

How can two courses disagree on something as basic as what goes in the numerator and denominator?! Can you resolve this paradox?

Hint 1: Units are not going to help you because c is available to convert length and time units.

Hint 2: What "group" of ideas are relevant when describing a particle? It might be a good idea to discuss this as a "group".

Coupled pendulum dispersion relation

So far we have only considered dispersion relations for waves in a continuum medium (string, water-surface, co-ax). Are there new features when the medium is **discrete**, like the torsionally coupled pendulums in assignment 2?

Here are the equations for the system of coupled pendulums you derived, in the small amplitude (small θ) limit:

$$\ddot{\theta}_n = -\omega_0^2\,\theta_n + \tau\,(\theta_{n-1} - 2\theta_n + \theta_{n+1})\ . \tag{1}$$

We've introduced two symbols for combinations of the original parameters:

$$\omega_0^2 = \frac{mgr}{I}, \qquad \tau = \frac{\kappa}{I}\ .$$

1. State in words what physical effects are quantified by ω_0 and τ.
2. Confirm that
$$\theta_n(t) = A\cos(\omega t)\cos(kn) \tag{2}$$
is a valid normal-mode solution of the equations (1) provided the normal-mode frequency has the following dispersion relation:
$$\omega(k) = \sqrt{\omega_0^2 + 2\tau(1-\cos k)}\ .$$
3. Here's how discreteness enters the analysis. Consider two k's that differ by an integer multiple of 2π, say k_1 and $k_2 = k_1 + 2\pi K$, where $K = 0, \pm1, \pm2$ etc. Show that the mode with parameter k_1 is exactly the same mode as the one with parameter k_2.
4. Since adding any integer multiple of 2π to k does not change the mode, we may restrict the range of k to be between $-\pi$ and π, because any other k gives the same mode as a mode in that range. Plot the dispersion relation $\omega(k)$ in that range of k. What is the minimum frequency, $\omega_{\rm min}$? What is the maximum frequency, $\omega_{\rm max}$?

Organ pipe, both ends open

This problem gives you another chance to prove you can solve standard normal mode problems. **To receive credit you need to clearly explain the steps to your solution**.

An organ pipe extends from $x = 0$ to $x = L$ and has both ends open. The pressure $p(x,t)$ in the pipe satisfies the wave equation with wave velocity v. Because of the open ends,

$$\frac{\partial p}{\partial x}(0,t) = 0 \ , \qquad \frac{\partial p}{\partial x}(L,t) = 0 \ .$$

1. Write down a general normal-mode solution.
2. What constraints do the two boundary conditions impose on your normal modes?
3. Express the possible normal mode frequencies in terms of the parameters v and L.

Point particle colliding elastically with a moving wall

A point particle, moving in any number of dimensions, maintains a constant velocity until it encounters a very massive moving wall. Model the collision with the wall by going into the frame co-moving with the wall. In this frame the wall is at rest and only the particle's velocity-component perpendicular to the wall is changed. The change is simply that the sign of the velocity component is flipped, since that preserves the kinetic energy (an elastic collision). Call x the coordinate perpendicular to the wall, and let V_x be the wall velocity and v_x the corresponding component of the particle velocity.

Your calculation involves three steps:

1. Find v'_x , the particle velocity in the wall's rest frame. You may use the Galilean transformation because $|V_x| \ll c$.
2. Obtain v''_x , the particle velocity in the wall's rest frame after the collision.
3. Finally, transform back to the "lab" frame to obtain the velocity v'''_x after the collision.